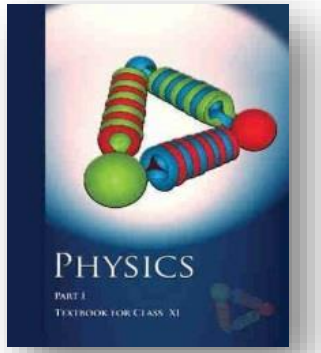

System of Particles and Rotational Motion

CHAPTER - 7

SYSTEM OF PARTICLES AND ROTATIONAL MOTION

Introduction

Centre of mass, Centre of Gravity

Motion of centre of mass

Linear momentum of a system of particles

Vector product of two vectors

Angular velocity and its relation with linear velocity, Kinematics of rotational motion about a fixed axis

Torque and angular momentum

Equilibrium of a rigid body

Moment of inertia

Theorems of perpendicular and parallel axes

Dynamics of rotational motion about a fixed axis

Angular momentum in case of rotations about a fixed axis

Rolling motion

Removed Topics

Theorems of Perpendicular and Parallel axes and their applications.

TOPICS

- Introduction
- **Center of mass
- Center of gravity
- Vector product
- Relation between Linear velocity and angular velocity
- **Conservation of Angular momentum
- Moment of Inertia
- **Rolling Motion
- **Rolling on inclined plane

INTRODUCTION

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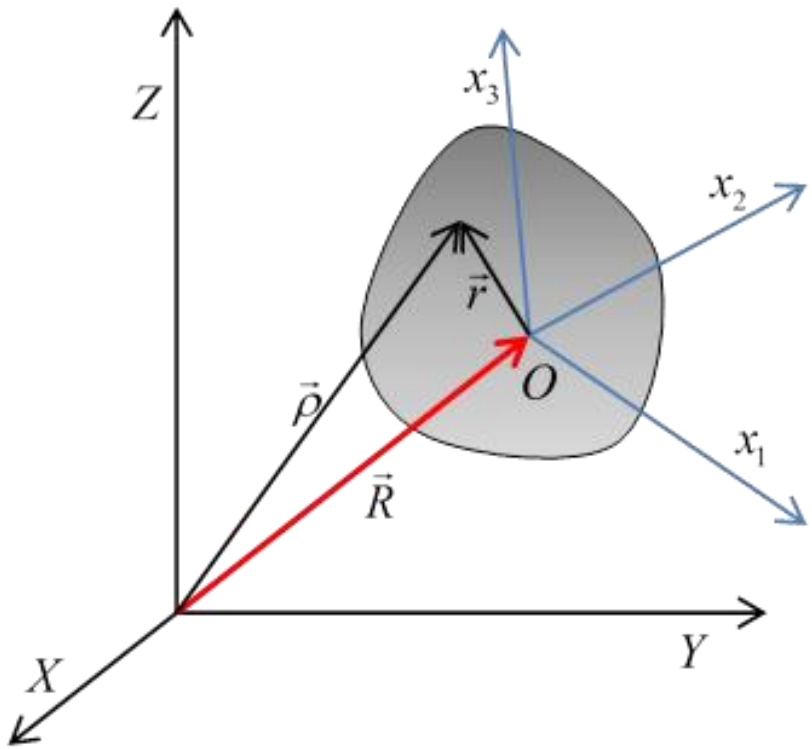
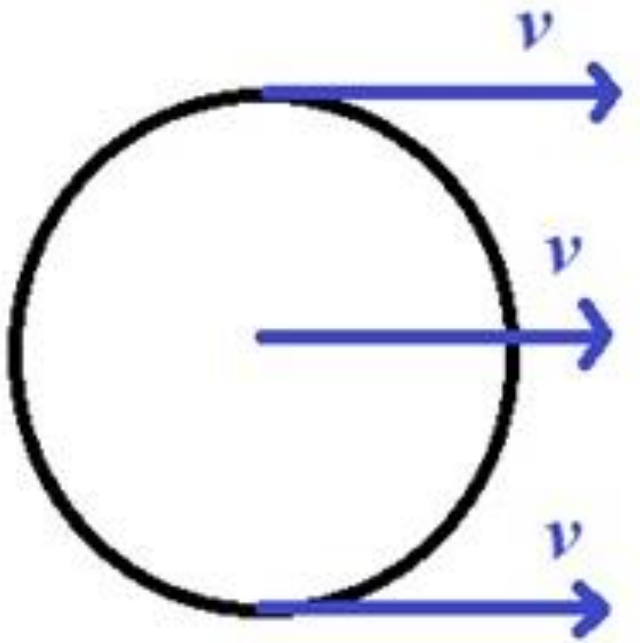
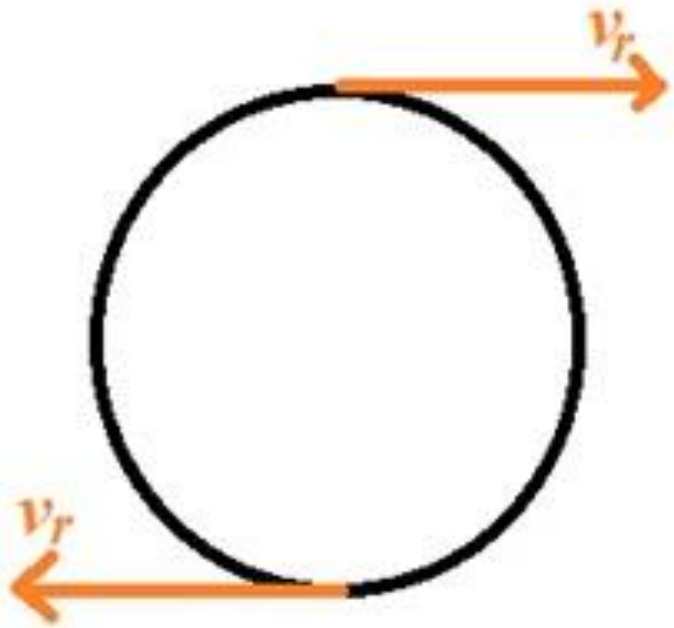
Rigid body : *It is a body with a perfetly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.*

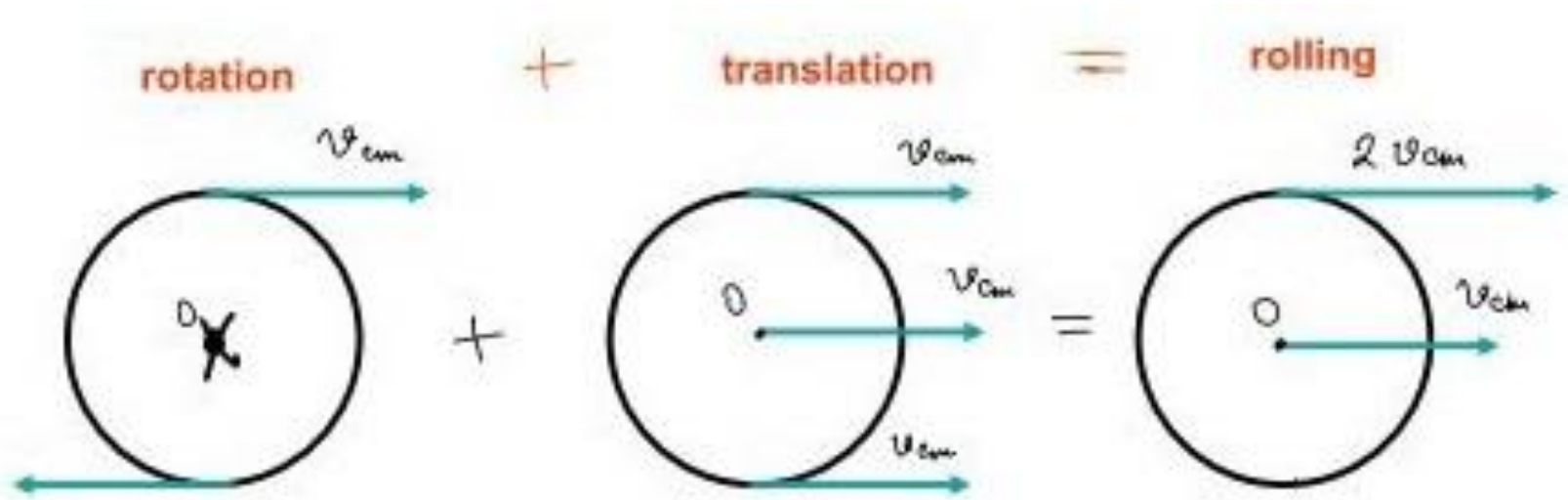




Translational



Rotational



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CENTER OF MASS

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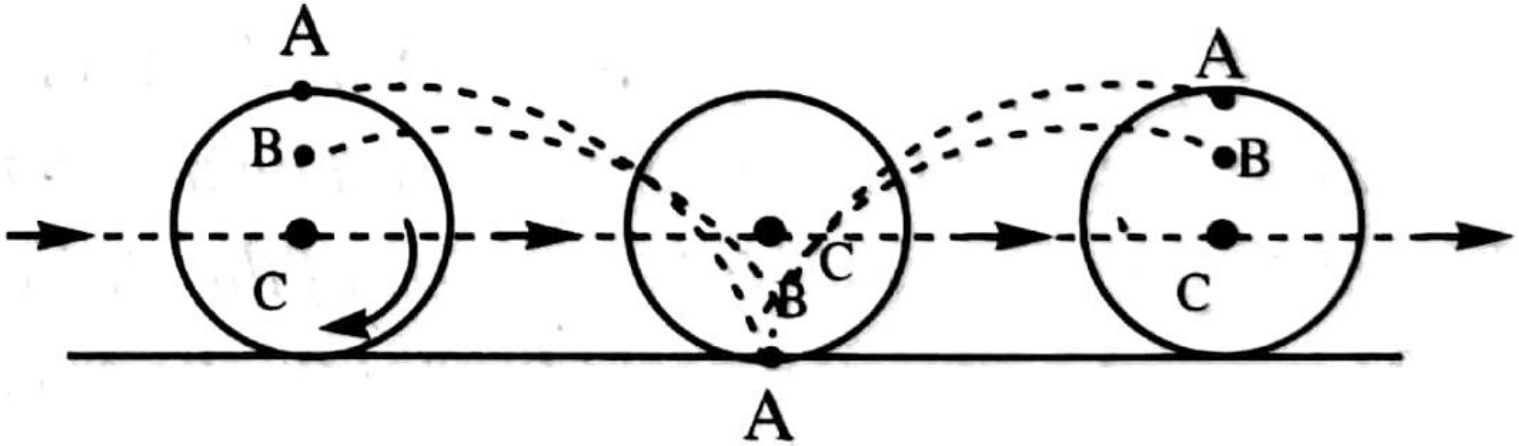
CENTRE OF MASS

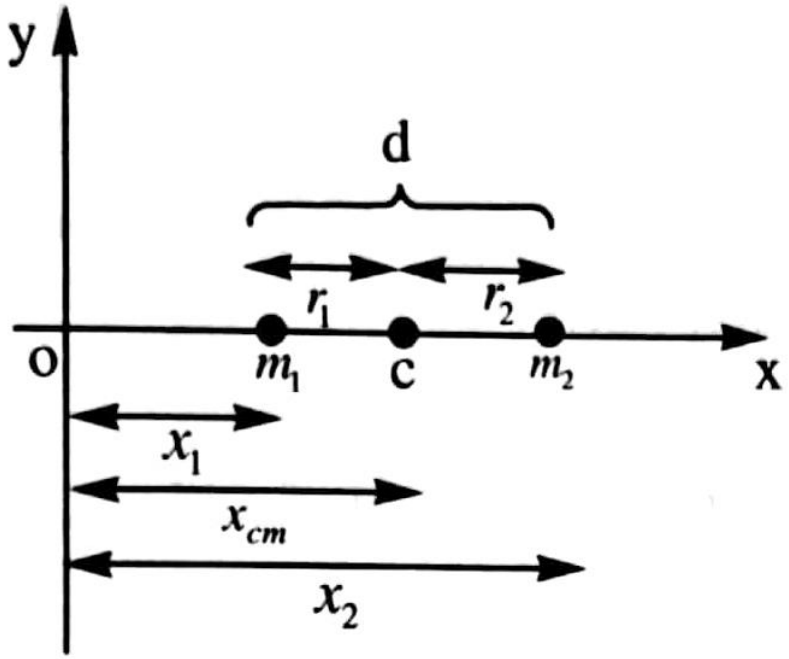
Centre of mass of a body or a system is a point where entire mass of the body or the system is supposed to be concentrated to describe its translatory motion and characterises the motion of whole body.



Position of Center of Mass:

$$r \alpha \frac{1}{m} \quad \Rightarrow \quad \frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$$\Rightarrow \quad m_1 r_1 = m_2 r_2$$



But from the above figure

$r_1 = x_{cm} - x_1$ and $r_2 = x_2 - x_{cm}$

$$\Rightarrow \quad m_1 [x_{cm} - x_1] = m_2 [x_2 - x_{cm}] \quad \Rightarrow \quad \boxed{x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

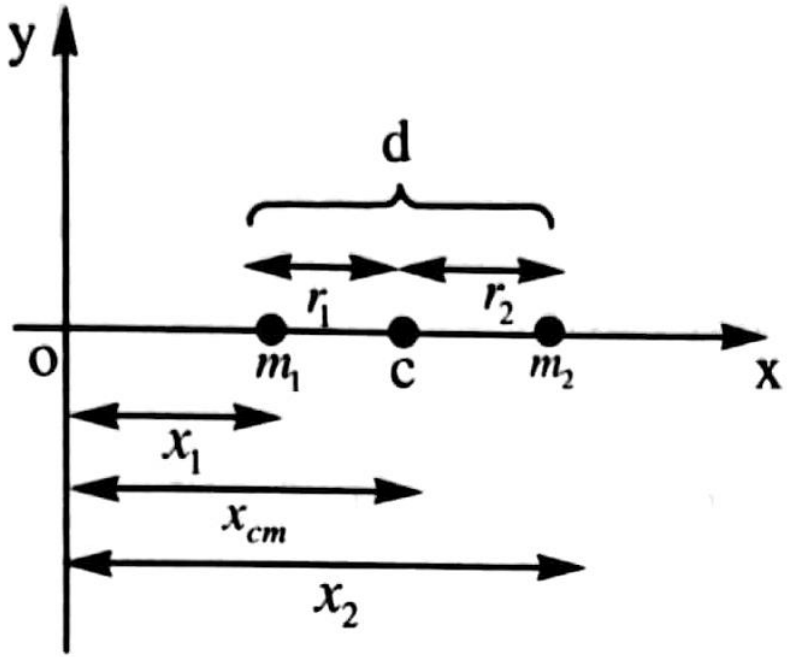
Also, the distance between particles

$$d = r_1 + r_2 \text{ and } r_2 = d - r_1$$

$$m_1 r_1 = m_2 [\, d - r_1 \,]$$

$$\Rightarrow r_1 = \boxed{\frac{m_2 d}{m_1 + m_2}}$$

$$r_2 = \boxed{\frac{m_1 d}{m_1 + m_2}}$$

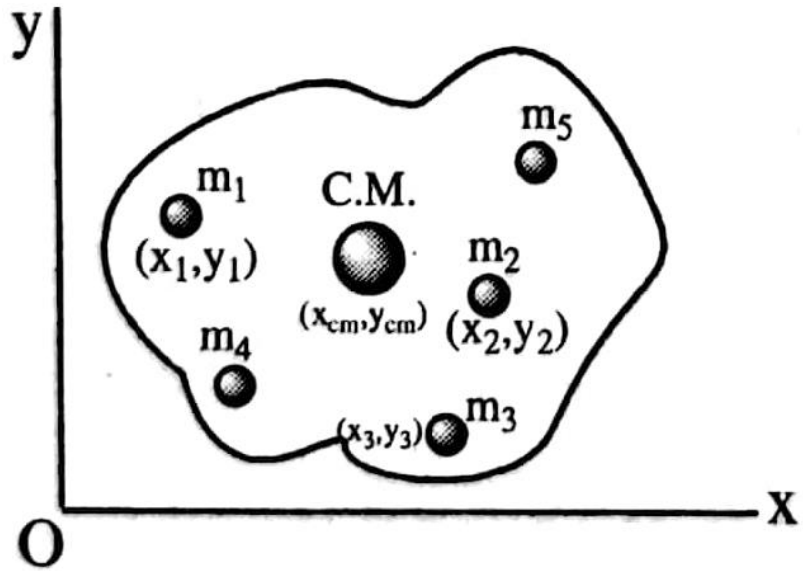


Center of mass of a system of particles in Plane:

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \ldots\ldots + m_nx_n}{m_1 + m_2 + \ldots\ldots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$$

and $$y_{cm} = \frac{m_1y_1 + m_2y_2 + \ldots\ldots + m_ny_n}{m_1 + m_2 + \ldots\ldots + m_n}$$

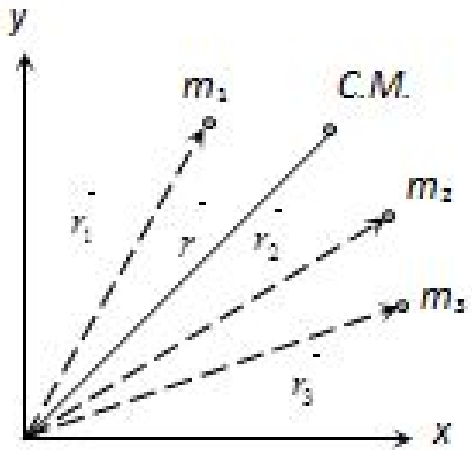
$$= \frac{\sum_{i=1}^{n} m_i y_i}{M}$$



$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots\ldots + m_n x_n}{m_1 + m_2 + \ldots\ldots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$$

$$x_{cm} = \frac{1}{M}\int x\, dm$$

$$\vec{r}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots\ldots + m_n \vec{r}_n}{m_1 + m_2 + \ldots\ldots m_n}$$



Velocity of centre of mass :

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ...m_n}$$

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{M}$$

$$\text{or} \quad M\vec{r}_{CM} = m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n$$

$$M\vec{r}_{CM} = m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n$$

$$M\frac{d\vec{r}_{CM}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + ...m_n\frac{d\vec{r}_n}{dt}$$

$$\therefore\ M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n$$

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n}{M}$$

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n}{M}$$

$$\text{or} \quad \vec{v}_{CM} = \frac{\sum_{i=1}^{n} m_i\vec{v}_i}{M}$$

For two particle system velocity of centre of mass

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

Momentum of centre of mass :

$$M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots + m_n\vec{v}_n$$

$$M\vec{v}_{CM} = \vec{p}_1 + \vec{p}_2 + \ldots\ldots\ldots + \vec{p}_n$$

$$\Rightarrow \vec{p}_{CM} = \sum_{i=1}^{n} \vec{p}_i$$

Acceleration of center of mass:

$$M\frac{d\vec{v}_{CM}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \ldots\ldots + m_n\frac{d\vec{v}_n}{dt}$$

$$M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots\ldots\ldots + m_n\vec{a}_n$$

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots\ldots + m_n\vec{a}_n}{M}$$

$$\vec{a}_{CM} = \frac{\sum_{i=1}^{n} m_i\vec{a}_i}{M}$$

$$\vec{F}_{CM} = \vec{F}_1 + \vec{F}_2 + \ldots\ldots\vec{F}_n$$

PROBLEM: ***When 'n' number of particles of masses m, 2m, 3m,..... nm are at distances $x_1=1, x_2=2, x_3=3, \ldots\ldots x_n=n$ units respectively from origin on the x-axis, then find the distance of centre of mass of the system from origin.***

Solution:

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \ldots\ldots + m_nx_n}{m_1 + m_2 + \ldots\ldots + m_n}$$

$$x_{cm} = \frac{m(1) + 2m(2) + 3m(3) + \ldots\ldots\ldots + (nm)n}{m + 2m + 3m + \ldots\ldots\ldots + nm}$$

$$x_{cm} = \frac{m(1^2 + 2^2 + 3^2 + \ldots\ldots + n^2)}{m(1 + 2 + 3 + \ldots\ldots + n)}$$

$$x_{cm} = \frac{m(1^2 + 2^2 + 3^2 + ... + n^2)}{m(1 + 2 + 3 + ... + n)}$$

$$X_{cm} = \frac{\left(\frac{n(n+1)(2n+1)}{6}\right)}{\left(\frac{n(n+1)}{2}\right)}$$

$$= \frac{2n+1}{3}$$

Characteristics of Center of Mass:

1. Total mass of the body seems to be concentrated at centre of mass.
2. The total external force (F_{ext}) applied on a body seems to be applied at centre of mass. $F_{ext} = M a_c$ where 'a_c' is acceleration of centre of mass.
3. Internal forces cannot change the motion of centre of mass.
4. A complex motion is a combination of translational and rotational motions. In complex motion centre of mass represents the entire translational motion of the whole body.
5. The momentum of a body is the product of mass of the body and velocity of centre of mass $\bar{P} = M V_c$
6. Co-ordinates of centre of mass do not depend on the co-ordinate system chosen.

CENTER OF GRAVITY

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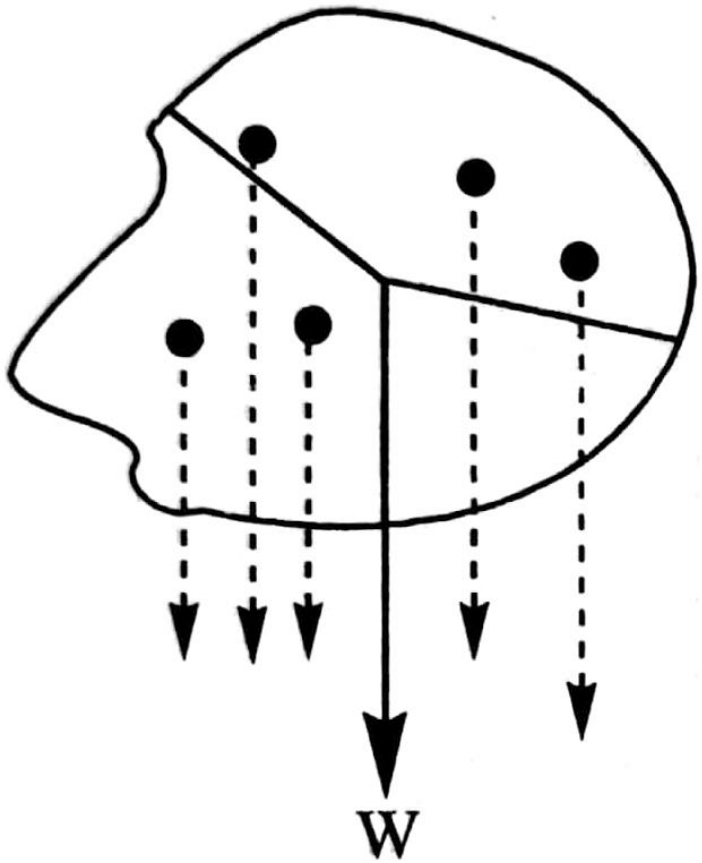
Center of Gravity:

The point, from where the resultant gravitational force of attraction or weight of the body acts is called centre of gravity.

(or)

The point in a body where it should be supported by an external force to keep the body in static equillibrium is called centre of gravity.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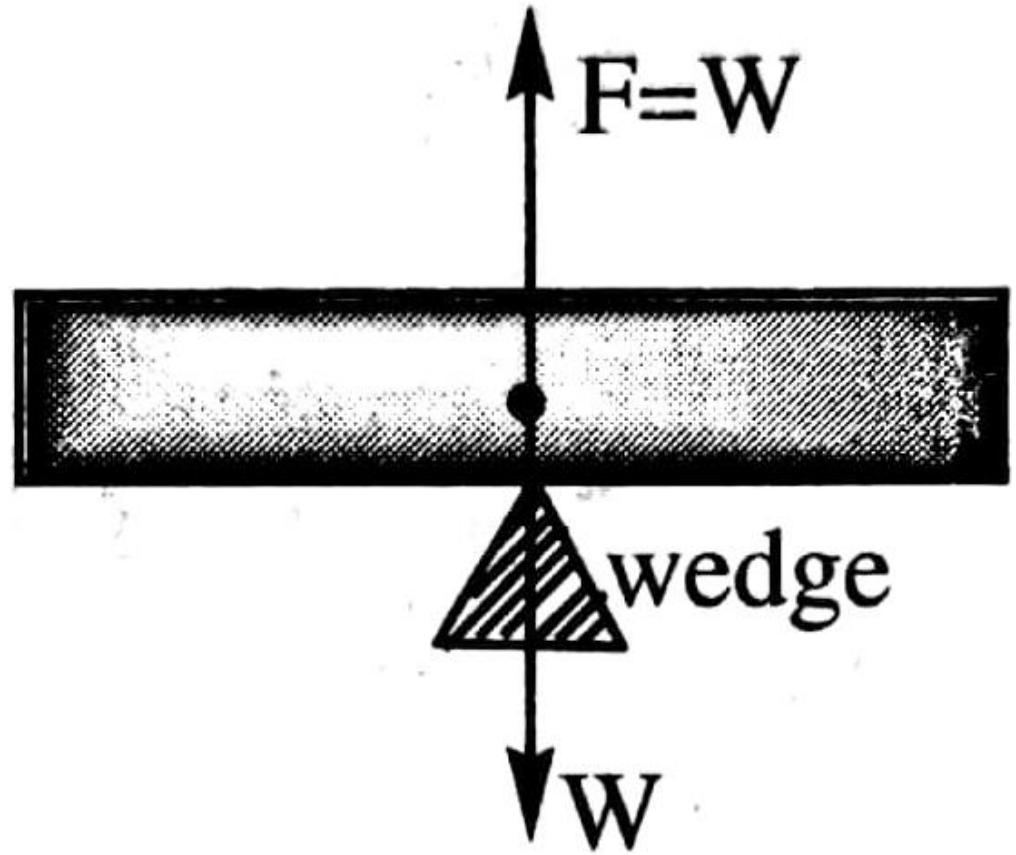
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Differences between Center of mass and Center of Gravity:

Centre of mass	Centre of gravity
1) A point inside a body at which the whole mass is supposed to be concentrated. A force applied at this point produces translatory motion.	1) A point inside a body through which the weight of the body acts.
2) It pertains to mass of the body.	2) It refers to weight acting on all particles of the body.
3) In case of small bodies centre of mass and centre of gravity coincide. (Uniform gravitational field)	3) In case of a huge body centre of mass and centre of gravity may not coincide. (Non uniform gravitational field)
4) Algebraic sum of moments of masses about centre of mass is zero.	4) Algebraic sum of moments of weights about centre of gravity is zero.
5) Centre of mass is used to study translatory motion of a body when it is in complicated motion.	5) Centre of gravity is used to know the stability of the body where it is to be supported.


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swaraj


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VECTOR PRODUCT

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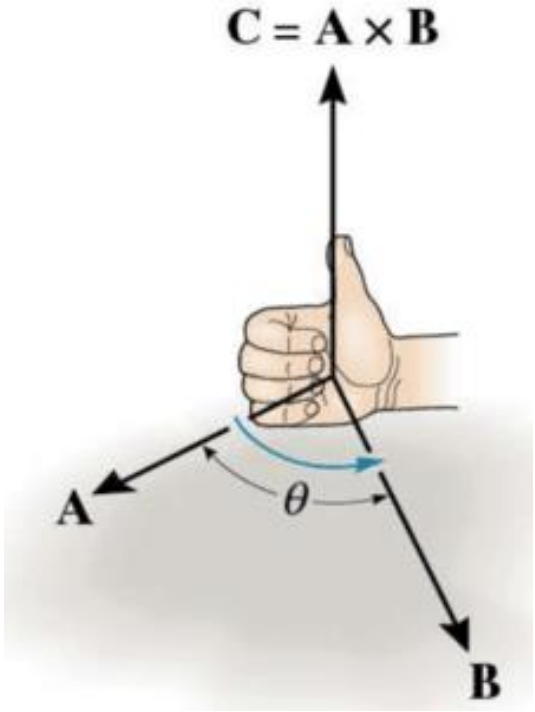
Vector product or Cross product:

It is defined as the product of the magnitude of vectors with sine of the angle between them.

$$\overline{A} \times \overline{B} = A\,B\,\mathbf{Sin}\theta\,\hat{n}$$

Resultant vector **C** perpendicular to the plane containing vectors **A** and **B**



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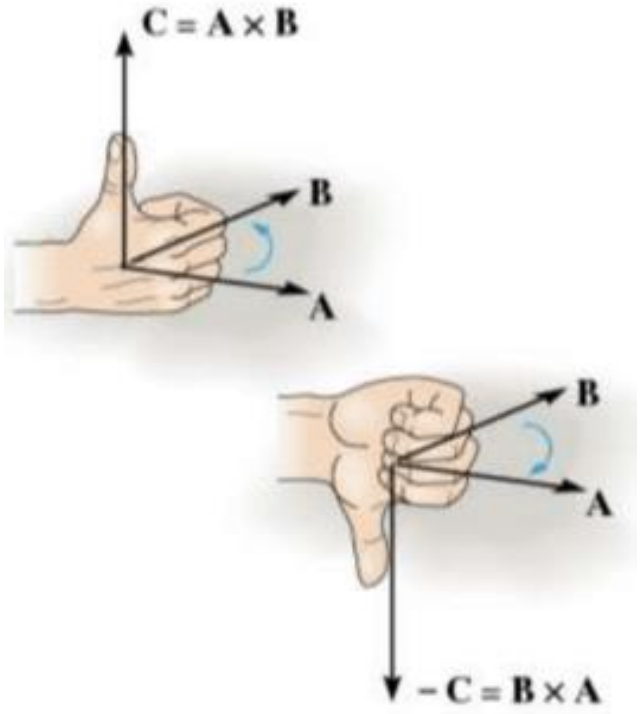
Properties of Cross Product:



(*i*) Vector product is anti-commutative

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(*ii*) Vector product is distributive over addition

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(*iii*) **Vector product of two parallel or antiparallel vectors is a null vector.**

$$\vec{A} \times \vec{B} = AB \sin (0° \text{ or } 180°) \hat{n} = \vec{0}$$

(*iv*) **Vector product of a vector with itself is a null vector.**

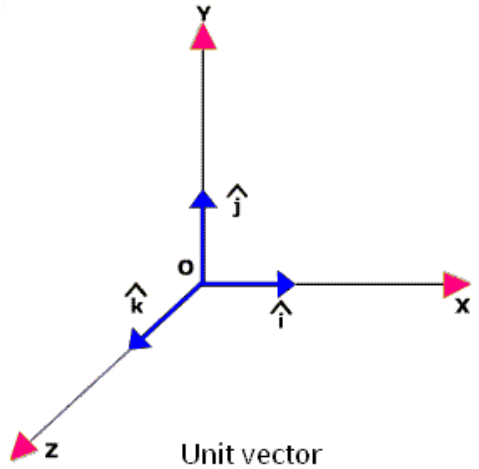
$$\vec{A} \times \vec{A} = AA \sin 0° \hat{n} = \vec{0}$$

(*v*) **The magnitude of the vector product of two mutually perpendicular vectors is equal to the product of their magnitudes.**

$$|\vec{A} \times \vec{B}| = AB \sin 90° = AB$$

(*vi*) **Vector product of orthogonal unit vectors.** The magnitude of each of the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ is 1 and the angle between any of two of them is 90°.

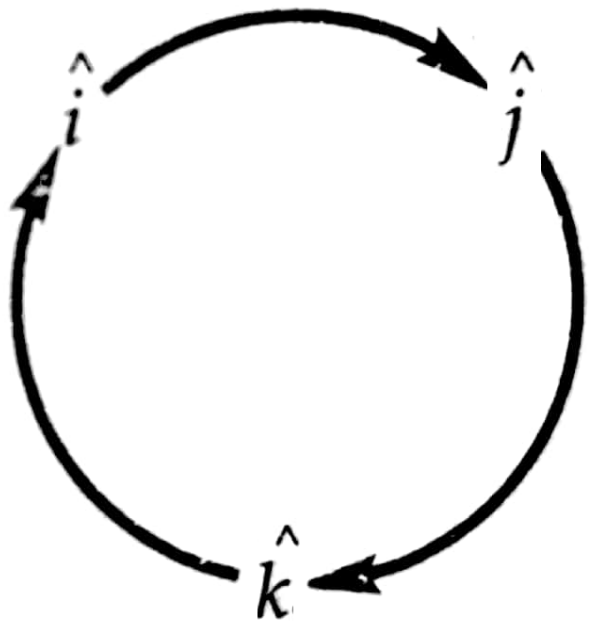
$$\therefore \quad \hat{i} \times \hat{j} = (1)(1) \sin 90° \, \hat{n} = \hat{n} = \hat{k}$$



$$\therefore \quad \hat{i} \times \hat{j} = \hat{k}$$

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$$



(*vii*) **The vector product of two vectors can be expressed in terms of their rectangular components as a determinant.**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Find the cross product of the vectors A = 3 i + 4 j and B = - 3 i + 7 j

Solution

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -3 & 7 & 0 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 4 & 0 \\ 7 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 4 \\ -3 & 7 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(21+12) = 33\,\hat{k}$$

Relation between Angular velocity and Linear velocity

Angular velocity (ω) :

Rate of change in angular displacement is called angular velocity.

Let angular displacement is $\Delta\theta$ over a time interval Δt

then average angular velocity

$$\omega = \frac{\Delta\theta}{\Delta t}, \quad \text{Unit} = \text{Radian / sec}$$

Angular acceleration (α) :

$$\text{Angular acceleration } \alpha = \frac{d\omega}{dt} \qquad \text{Unit : Radian /sec}^2$$

Uniform Circular motion:

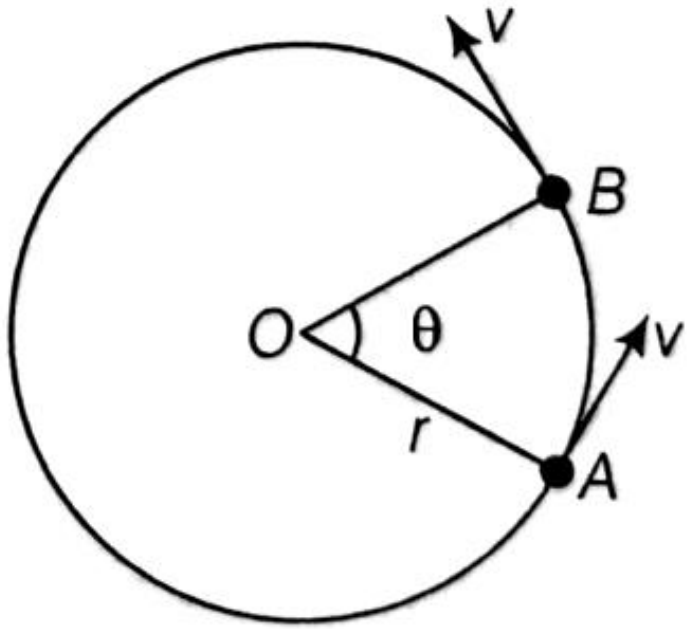
Relation between angular velocity, frequency and time period:

If time $t = T$, $\theta = 2\pi$ radian

$$\text{Angular velocity, } \omega = \frac{\theta}{t}$$

$$= \frac{2\pi}{T}$$

$$= 2\pi f$$



Relation between angular velocity and linear velocity:

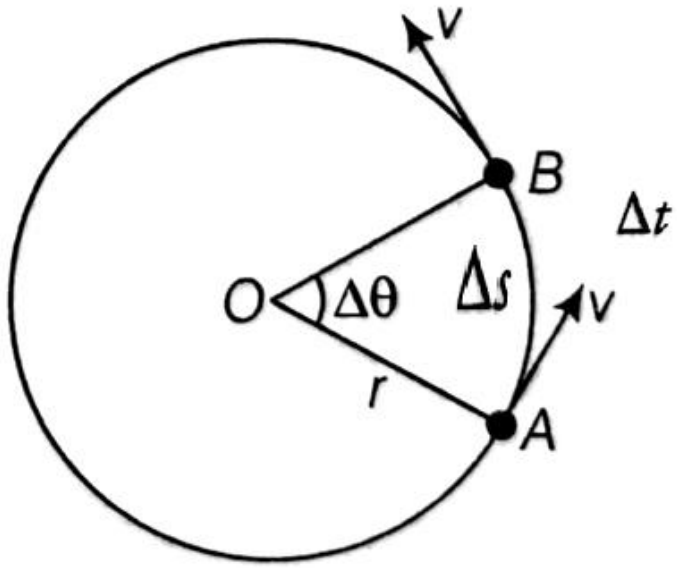
Angular velocity $\omega = \frac{\Delta\theta}{\Delta t}$ ----- 1

Linear velocity $v = \frac{\Delta s}{\Delta t}$ ----- 2

$\Delta s = r\,\Delta\theta$ ----- 3

Substituting eqn 3 in eqn 2 $v = r\frac{\Delta\theta}{\Delta t} = r\,\omega$

From eqn 1



Translatory Motion
(When linear acceleration is constant)

1) $v = u + at$

2) $S = ut + \frac{1}{2}at^2$

3) $v^2 - u^2 = 2aS$

4) $S_n = u + \frac{a}{2}(2n-1)$

Circular Motion
(When angular acceleration is constant)

1) $\omega_2 = \omega_1 + \alpha t$

2) $\theta = \omega_I t + \frac{1}{2}\alpha t^2$

3) $\omega_2^2 - \omega_1^2 = 2\alpha\theta$

4) $\theta_n = \omega_1 + \frac{\alpha}{2}(2n-1)$

Conservation of Angular Momentum

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Moment of force couple or torque (τ):

The moment of force is called torque or moment of force couple.

$$\text{Torque } \tau = \bar{r} \times \bar{F} = |\bar{r}| \times |\bar{F}| \sin\theta \cdot \vec{n}$$

Unit : Newton metre (N - m)

Angular momentum (L):

$$\bar{L} = \bar{r} \times \bar{p} = |\bar{r}||\bar{p}| \sin\theta \cdot \hat{n} \qquad L = I\omega$$

Torque and angular momentum :

The time rate of change of the angular momentum of a particle is equal to torque acting on it.

$$\text{Torque } \tau = \frac{d\bar{L}}{dt} = \bar{r} \cdot \frac{d\bar{p}}{dt}$$

Law of conservation of angular momentum :

$$\text{When } \tau_{ext} = 0 \text{ then } \frac{d\bar{L}}{dt} = 0$$

When external torque (τ_{ext}) is zero, the total angular momentum of a system is conserved i.e., it remains constant.

i.e. angular momentum $\bar{L}$ is constant.

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

$\frac{d\omega}{dt}$ is the rate of change of angular velocity called angular acceleration (α)

∴ The relation between Torque and angular acceleration is,

$$\tau = I\alpha$$

Law of conservation of angular momentum:

If external torque $\tau = 0$

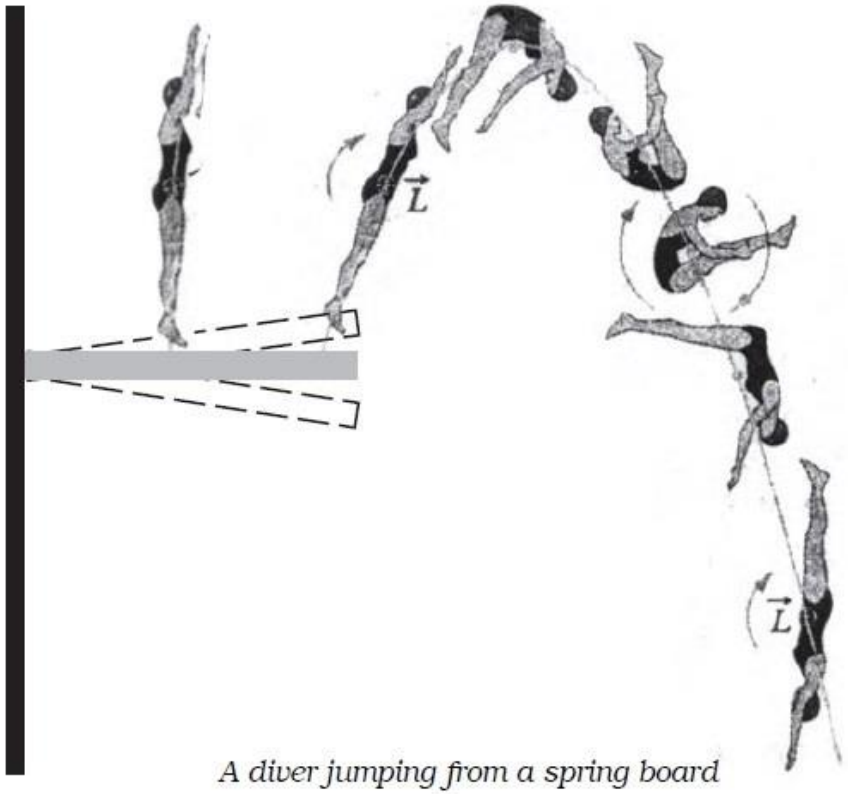
$$\text{then } \frac{d\vec{L}}{dt} = \tau = 0$$

$\therefore$ Change in angular momentum is also zero.

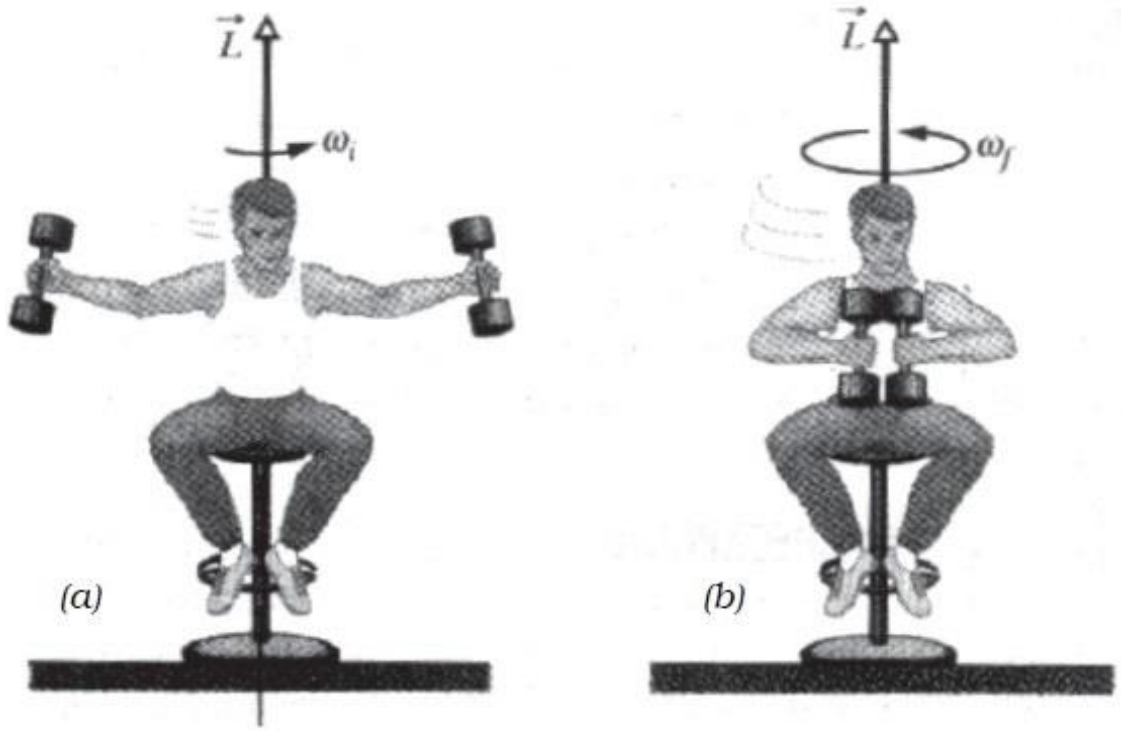
$$\Rightarrow \quad \vec{L}_2 - \vec{L}_1 = 0$$

$$\text{i.e., } \quad I_2\omega_2 - I_1\omega_1 = 0 \qquad \text{or} \qquad I_1\omega_1 = I_2\omega_2$$

$$I_1\omega_1 = I_2\omega_2$$



A diver jumping from a spring board



A person rotating on a turn table

Image source: https://engineersfield.com/

Equilibrium of a rigid body:

if both its linear momentum and angular momentum are not changing with time.

The vector sum of all the forces acting on a rigid body must be zero.

$$\text{i.e. } F_1 + F_2 + \ldots\ldots\ldots + F_n = \sum_{i=1}^{n} F_i = 0$$

This condition provides translational equilibrium of the body.

The vector sum of all the torques acting on a rigid body must be zero

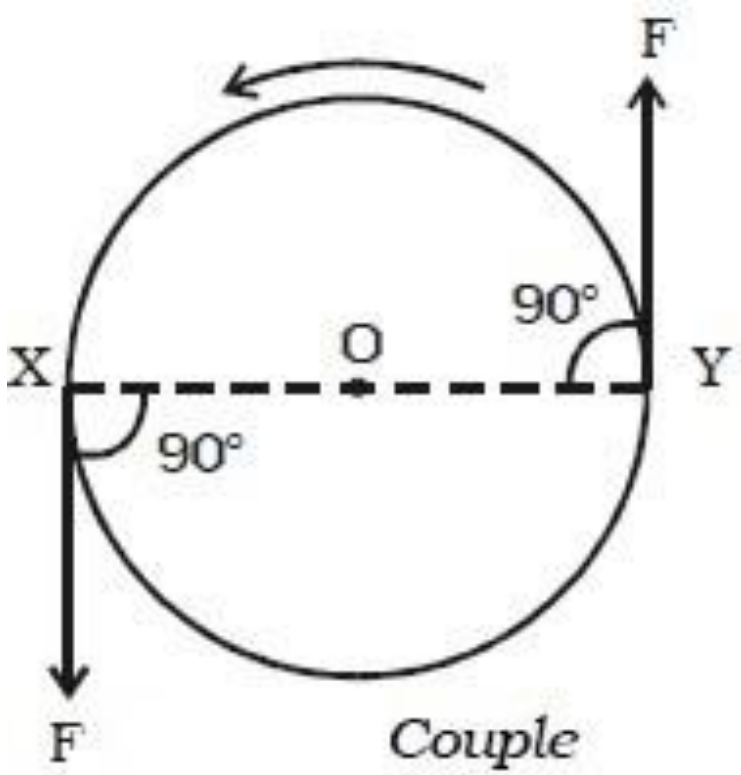
$$\text{i.e. } \tau_1 + \tau_2 \ldots\ldots\ldots + \tau_n = \sum_{i=1}^{n} \tau_i = 0$$

This condition provides rotational equilibrium of the body.

Couple (or) Force couple :

A pair of equal and opposite forces with different lines of action is known as couple.

A couple produces rotation without translation.



Couple

Principles of moments :

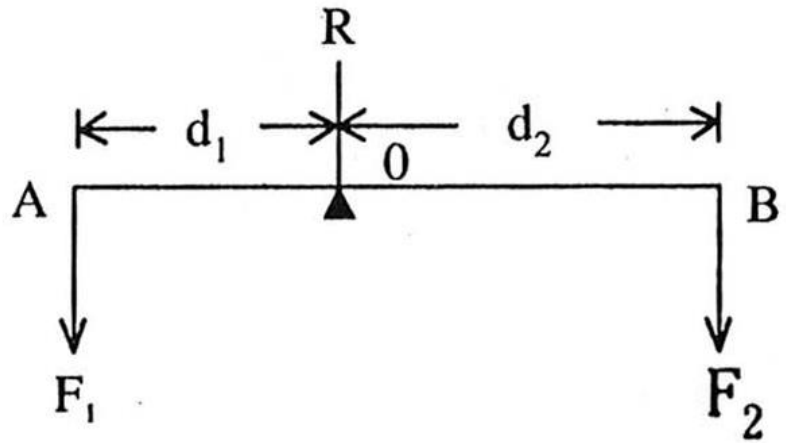
For a lever to be in mechanical equilibrium let R is the reaction of the support at fulcrum. It is directed upwards and F_1 and F_2 are the forces then

For translational equilibrium $R - F_1 - F_2 = 0$

i.e. alzebraic sum of forces must be zero.

For rotational equilibrium $d_1F_1 - d_2F_2 = 0$.

i.e. alzebraic sum of moments must be zero.



Moment of Inertia

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Moment of Inertia

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Moment of Inertia (I) :

The inertia of a rotating body is called moment of inertia.

Mathematically, Moment of Inertia, $I = \sum_{i=1}^{n} m_i r_i^2 = MR^2$

Radius of gyration (k) :

Radium of gyration of a body about an axis may be defined as the distance from the axis of a mass point whose mass is equal to whole mass of the body and whose moment of inertia is equal to moment of inertia of the whole body about that axis.

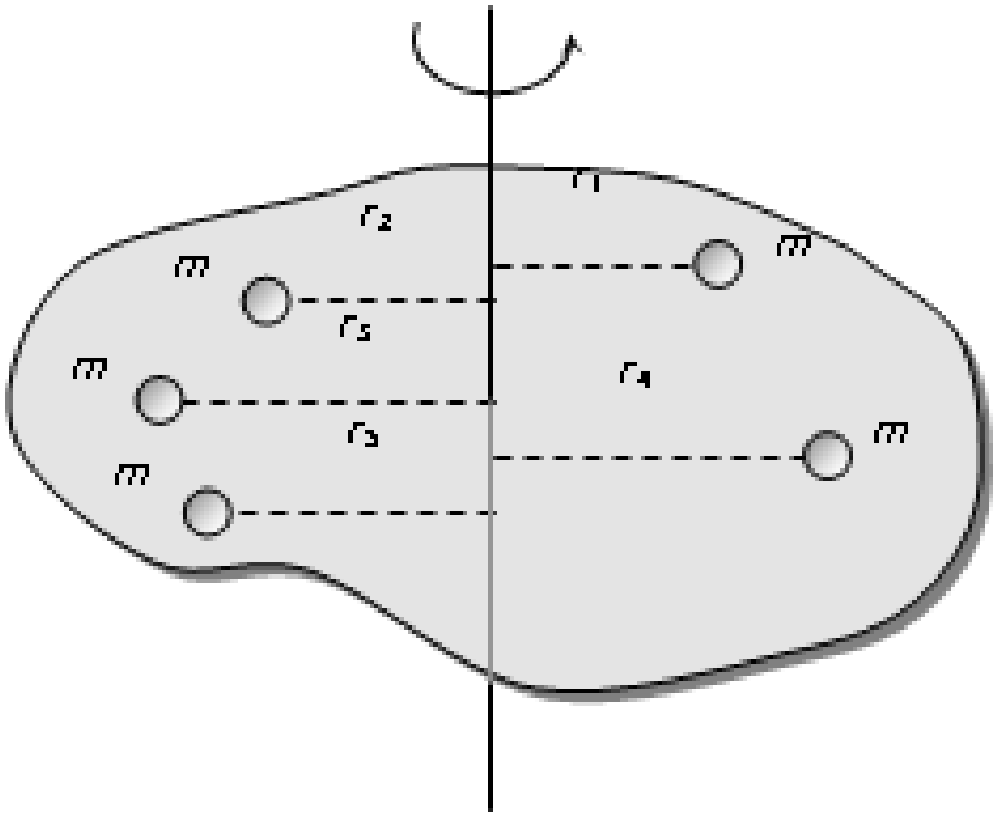
$$I = mr_1^2 + mr_2^2 + mr_3^2 + \cdots + mr_n^2$$

$$= m\left(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2\right)$$

$$= nm\left(\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}\right)$$

$$I = MK^2$$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}}$$



Rolling Motion

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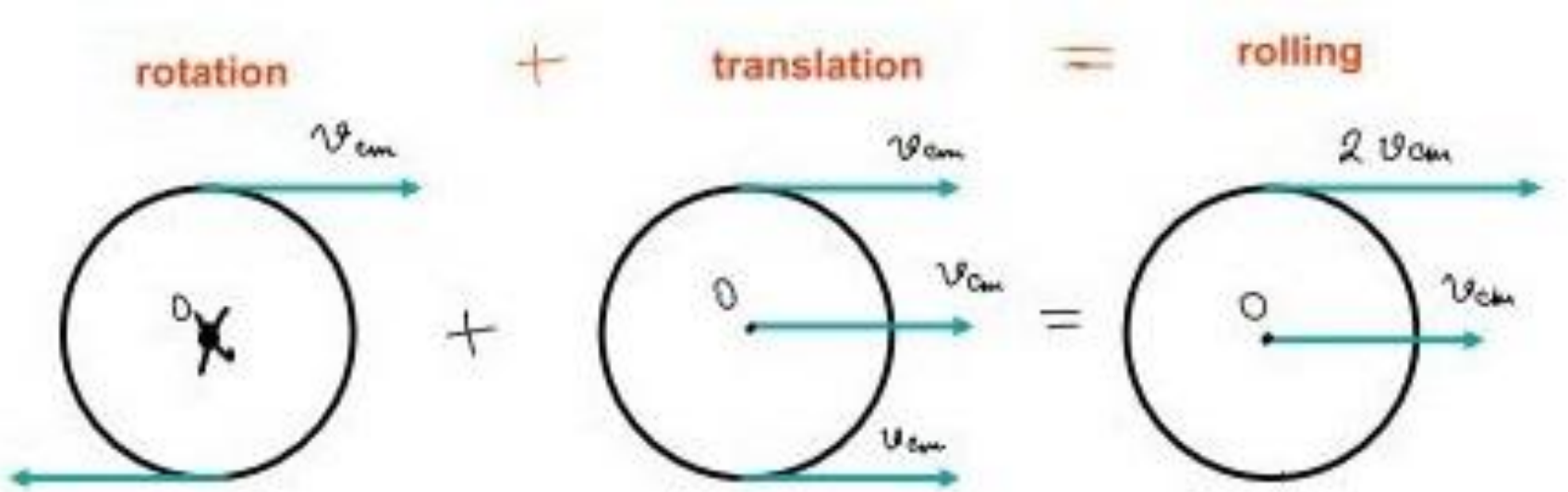
Rolling Motion:



Final velocity and total energy of a rolling body without slipping:

A rolling body has both translational kinetic energy and rotational kinetic energy.

So the total K.E energy of a rolling body is,

$$K.E_T = K.E_{(Translational)} + K.E_{(Rotational)}$$

$$\Rightarrow \quad K.E_T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$\Rightarrow \quad K.E_T = \frac{1}{2}mv^2 + \frac{1}{2}mk^2\frac{v^2}{r^2} \qquad \left[\because I = mk^2 \text{ and } v = r\omega\right]$$

$$\Rightarrow \quad K.E_T = \frac{1}{2}mv^2 + \frac{1}{2}mk^2\frac{v^2}{r^2} \qquad \left[\because I = mk^2 \text{ and } v = r\omega\right]$$

$$\text{Total energy } E = \frac{1}{2}\,mv^2\left[1+\frac{k^2}{r^2}\right]$$

$$\Rightarrow \quad v^2 = \frac{2E}{m\left[1+\frac{k^2}{r^2}\right]}$$

$$\therefore \quad \text{Velocity of the body, } v = \sqrt{\frac{2E}{m\left[1+\frac{k^2}{r^2}\right]}}$$

Rolling on Inclined Plane

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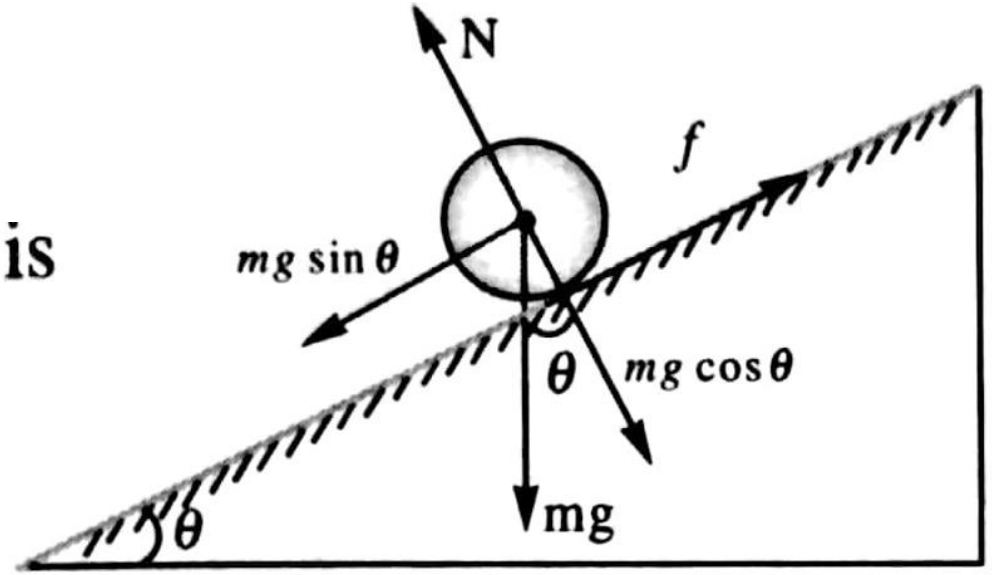
Rolling a body on an Inclined Plane:

$$N = mg\cos\theta$$

The net force acting down the inclined plane is



$$F_R = mg\sin\theta - f_s$$

$$\therefore a = \frac{mg\sin\theta - f_s}{m}$$

$$a = g\sin\theta - \frac{f_s}{m}$$

$$\tau = f_s R \qquad \alpha = \frac{f_s R}{I}$$

But $\alpha = \frac{a}{R}$ and $I = mk^2$

$$\therefore \frac{a}{R} = \frac{f_s R}{mK^2}$$

$$\Rightarrow a = \frac{f_s R^2}{mK^2} \Rightarrow f_s = \frac{mK^2 a}{R^2}$$

$$f_s = \frac{mK^2a}{R^2}$$

$$a = g\sin\theta - \frac{f_s}{m}$$

$$a = g\sin\theta - \frac{\cancel{m}K^2a}{R^2\cancel{m}}$$

$$a = g\sin\theta - \frac{K^2a}{R^2}$$

$$\therefore\ a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

$$a = \frac{g\sin\theta}{1+\beta}$$

Velocity of the rolling body (without slipping) on reaching the bottom of the inclined plane:

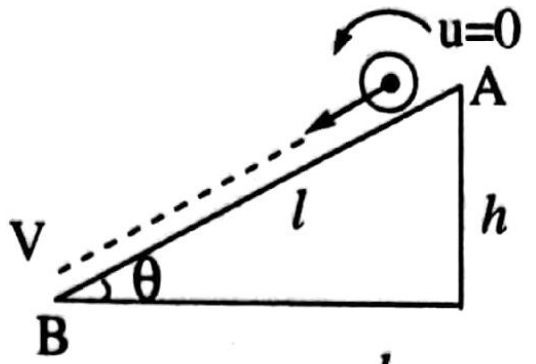
$$PE_A = KE_B$$

$$mgh = \frac{1}{2}mv^2[1+\beta]$$

$$\Rightarrow \boxed{v = \sqrt{\frac{2gh}{1+\beta}}}$$

$$\boxed{v = \sqrt{\frac{2gl\sin\theta}{1+\beta}}}$$

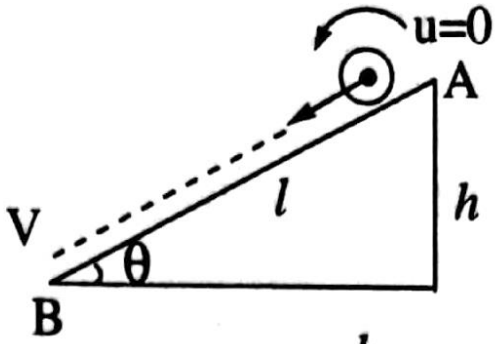
$$v^2 = \frac{2E}{m\left[1+\frac{k^2}{r^2}\right]}$$



$$\sin\theta = \frac{h}{l}$$

$$\Rightarrow h = l\sin\theta$$

Time taken by the rolling body to reach the bottom of the plane:



$$l = ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2l}{a}} \qquad a = \frac{g\sin\theta}{1+\beta}$$

$$\sin\theta = \frac{h}{l}$$

$$\Rightarrow h = l\sin\theta$$

$$t = \sqrt{\frac{2l(1+\beta)}{g\sin\theta}} \quad \Rightarrow \quad t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}(1+\beta)}$$

$$l = \frac{h}{\sin\theta}$$

$$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}(1+\beta)}$$

For sliding bodies $\beta = 0$,

$$\Rightarrow \quad t = \sqrt{\frac{2l}{g\sin\theta}} \quad = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$$

THANK YOU

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