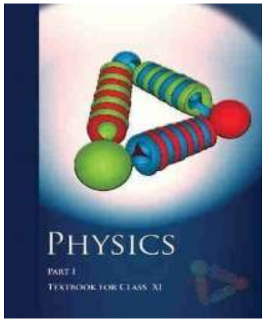


Motion in a Plane

CHAPTER - 4

MOTION IN A PLANE

Introduction

Scalars and vectors

Multiplication of vectors by real numbers

Addition and subtraction of vectors. graphical method

Resolution of vectors

Vector addition. analytical method

Motion in a plane

Motion in a plane with constant acceleration

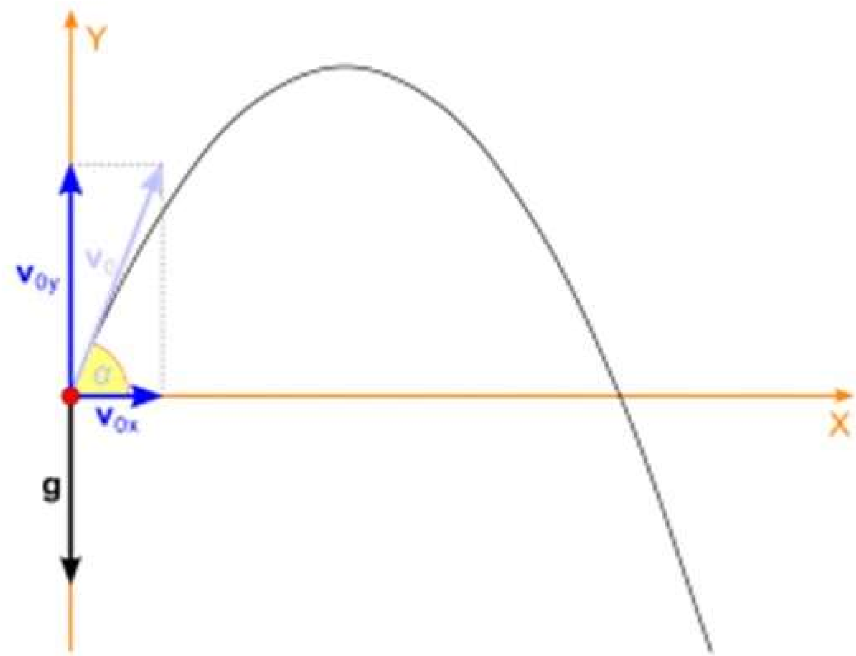
Relative velocity in two dimensions

Projectile motion

Uniform circular motion

Nothing is removed from this Chapter

INTRODUCTION


Y
v_{0y}
v_0
α
v_{0x}
X
g

Scalars Or Scalar Quantities:

The physical quantities which have only magnitude but no direction are called scalar quantities.

Eg:

Temperature, Mass, Distance, Time, Work, Power, etc.,

Vectors Or Vector Quantities:

The physical quantities which have magnitude and directions are called scalar quantities.

Eg:

Displacement, Velocity, Acceleration, Force, etc.,

TYPES OF VECTORS

Polar Vectors:

Vectors which have a starting point or a point of application are called polar vectors.

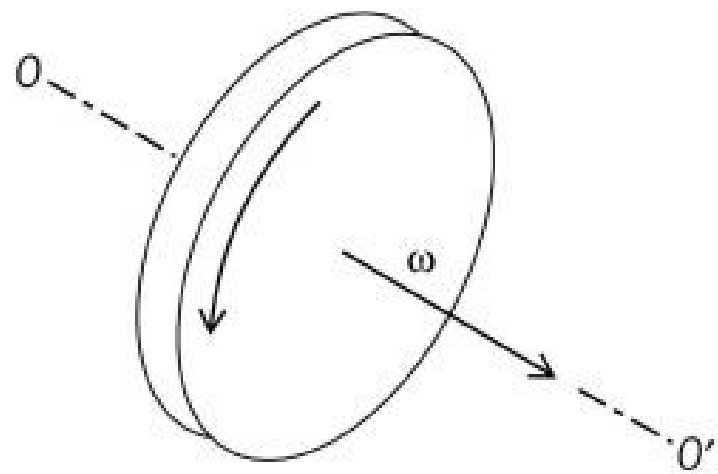
Eg:

Displacement, Force, etc.,

Axial Vectors:

Vectors which represents the rotational effect and acts along the axis of rotation are called axial vectors.

Eg: Angular velocity, Angular momentum, Torque, etc.,



Null Vector:

A vector with magnitude zero and having an arbitrary direction is called a null vector.

Equal Vectors:

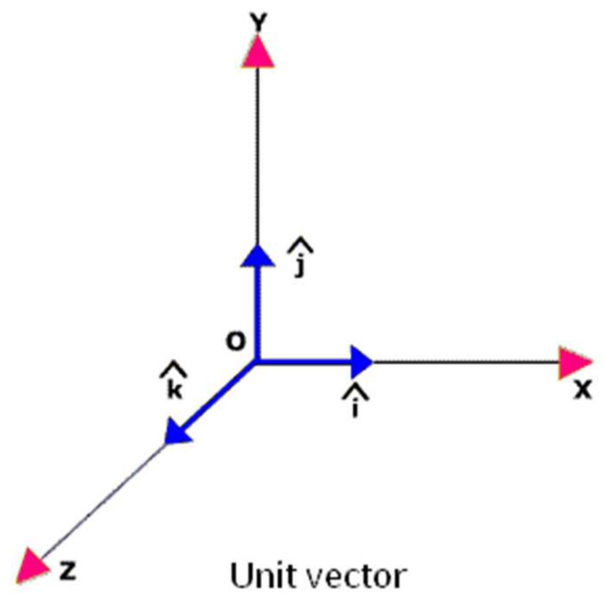
Two vectors are said to be equal if they have equal magnitude and same direction

Unit Vector:

A vector having magnitude equal to unity but having a specific direction is called a unit vector.

If **A** is a vector then unit vector is represented as $\hat{A}$

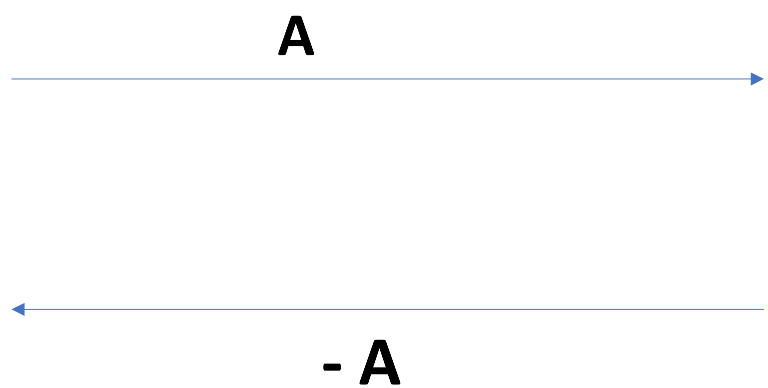
$$\hat{A} = \frac{\bar{A}}{A}$$



Negative Vector:

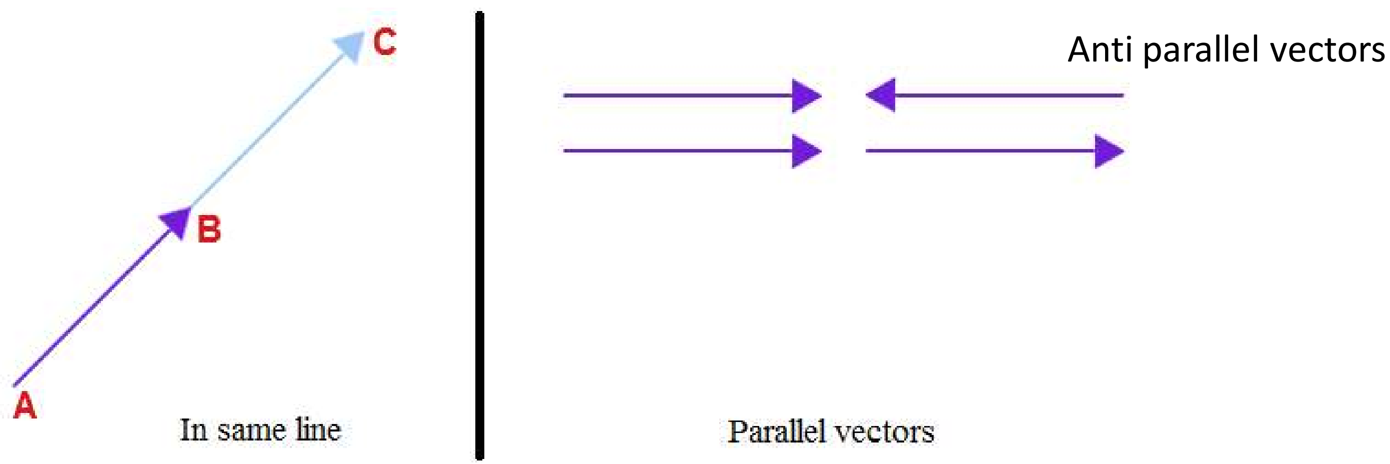
Two vectors are said to be the negative of each other if their magnitudes are equal but directions are opposite.

The negative vector of **A** is represented as - **A**



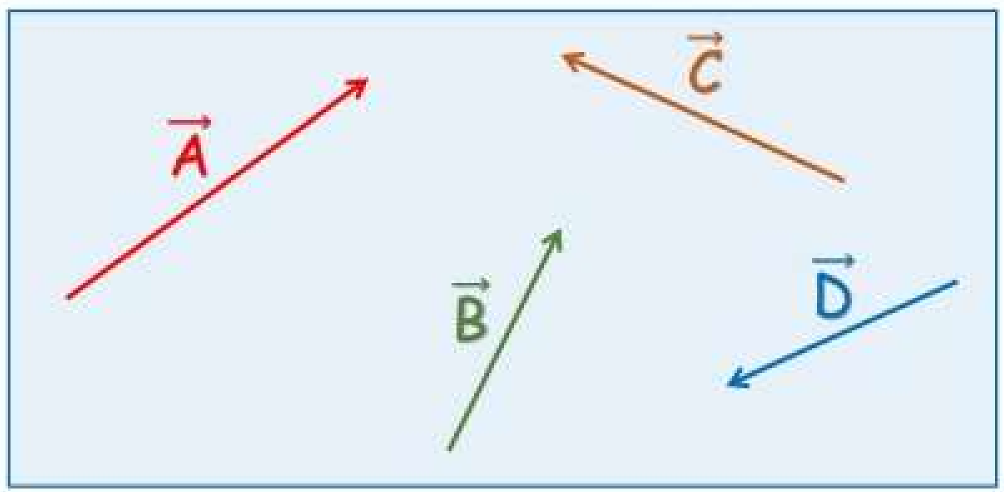
Collinear Vectors:

Two vectors are said to be collinear vectors, if they act along a same line.



Coplanar Vectors:

The vectors lying in the same plane are called coplanar vectors.



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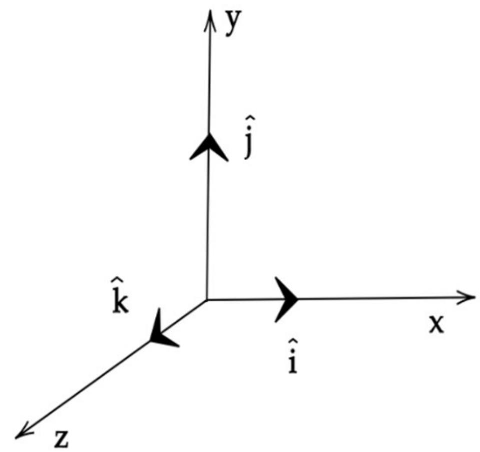
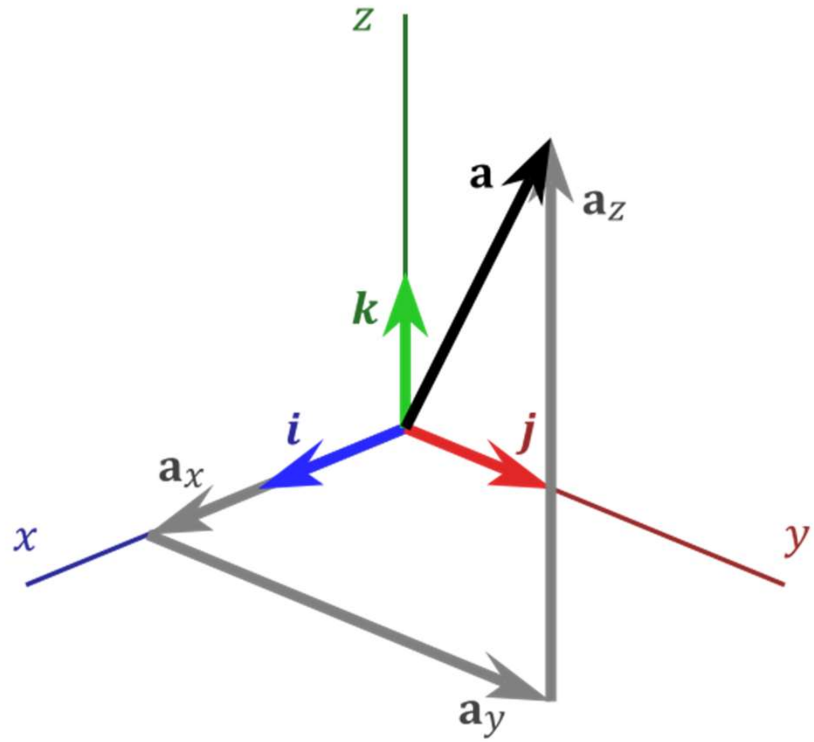
Orthogonal Vectors:

If the vectors are perpendicular to each other, they are known as orthogonal unit vectors.



Vectors representation:

$$\overline{a} = a_x\,\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$



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$$\cos\theta = \frac{x}{v}$$

$$\Rightarrow x = v\cos\theta$$

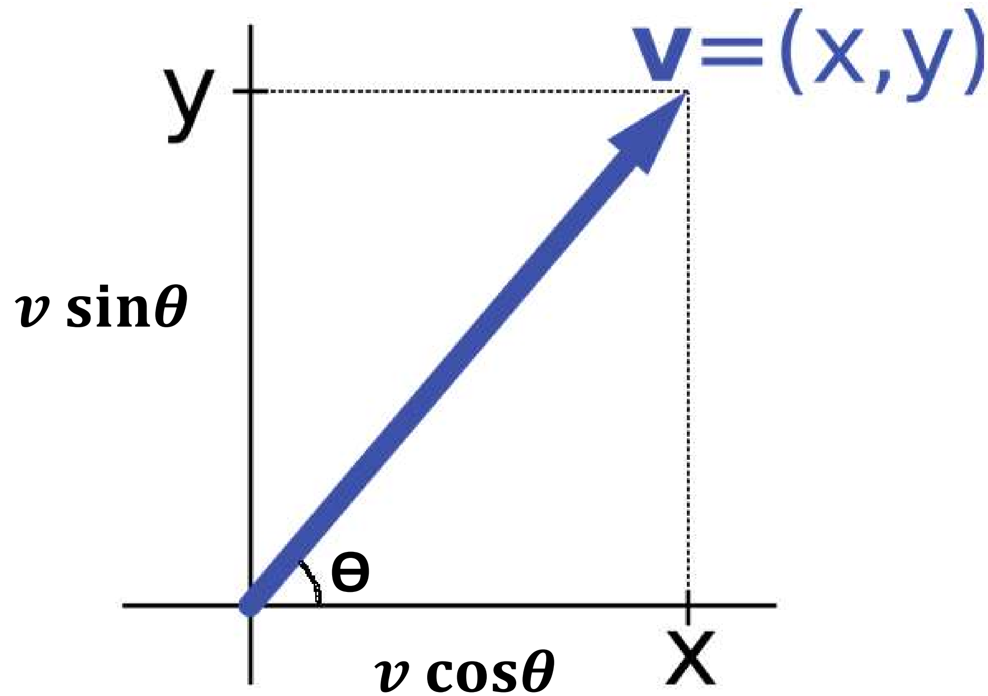
$$\sin\theta = \frac{y}{v}$$

$$\Rightarrow y = v\sin\theta$$

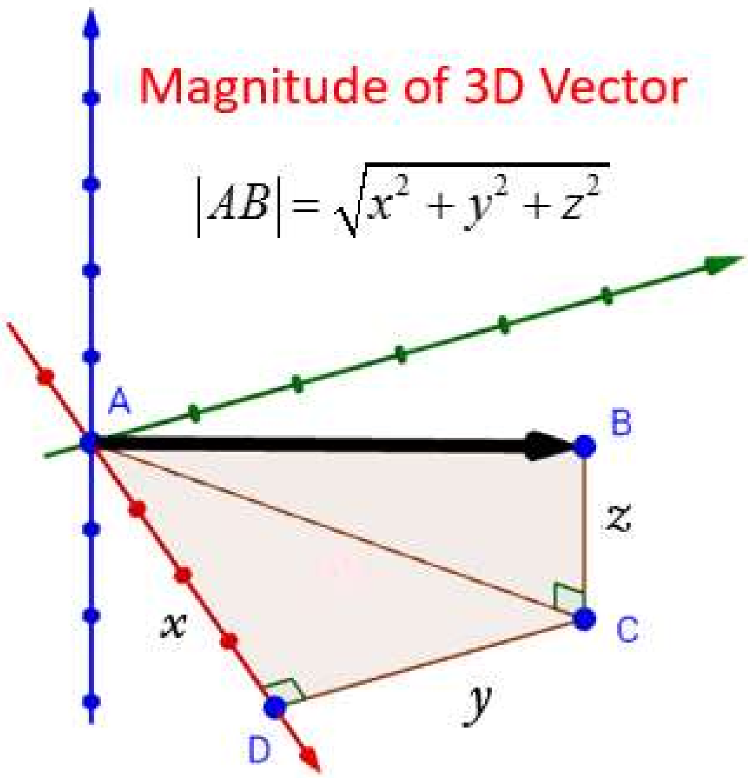
Magnitude of vector

$$v = \sqrt{x^2 + y^2}$$

Magnitude of vector:



$$\bar{v} = x\,\hat{i} + y\,\hat{j}$$



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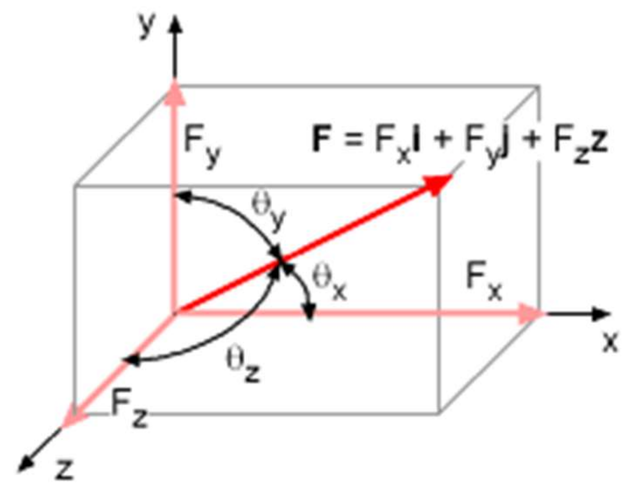

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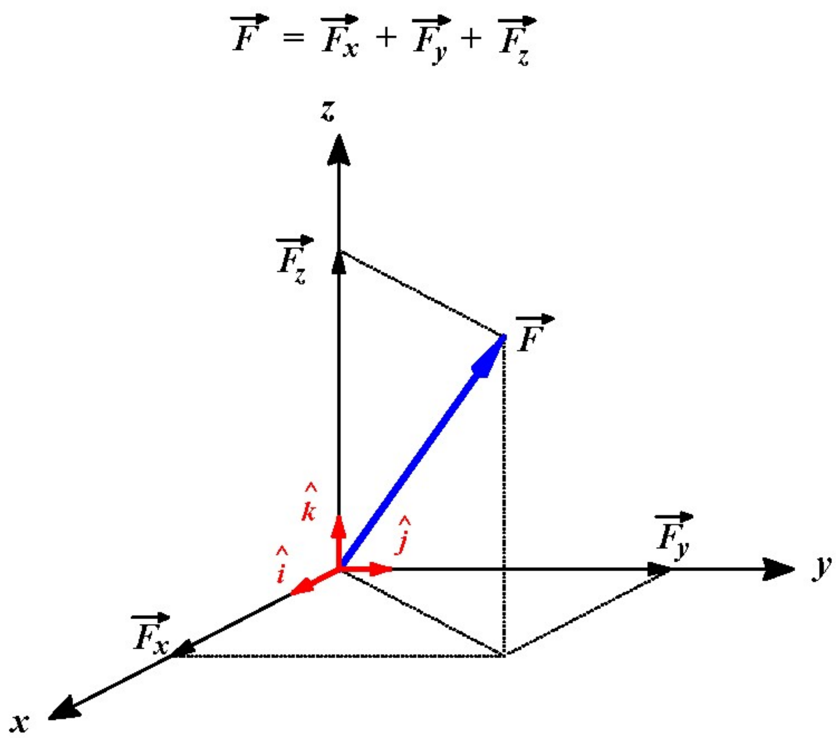

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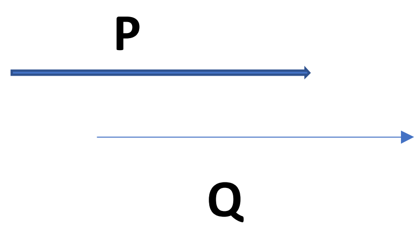
$$\bar{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

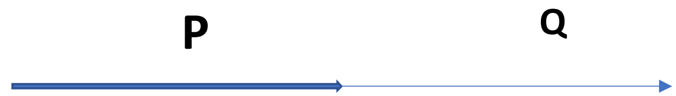
$$F = |\bar{F}| = \sqrt{F_x{}^2 + F_y{}^2 + F_z{}^2}$$

$$If\ \ \bar{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\ \ then$$

$$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$$

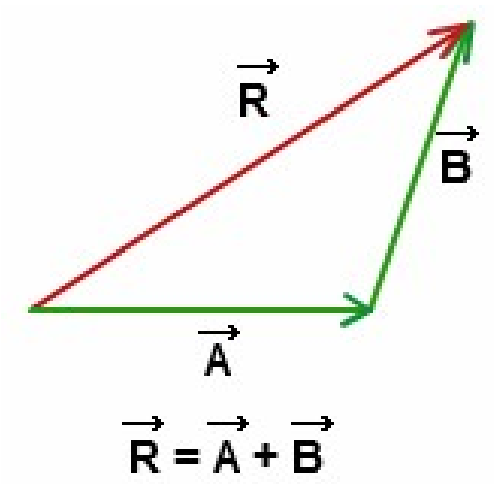
Addition of Vectors:



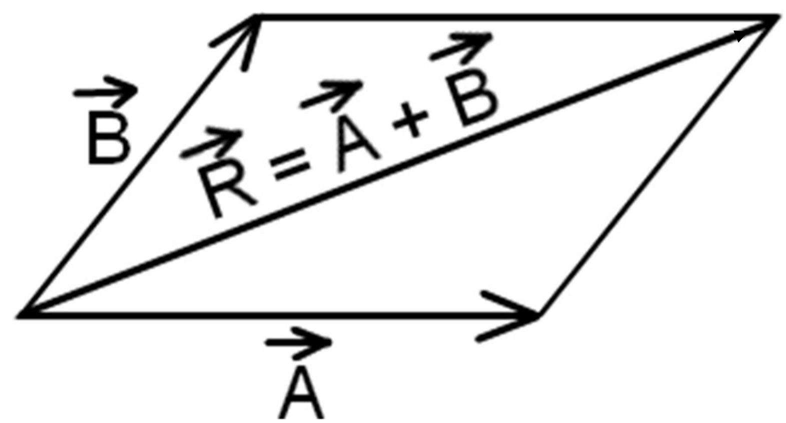


$$R = P + Q$$

Triangle law of vectors:



Parallelogram law of vectors:

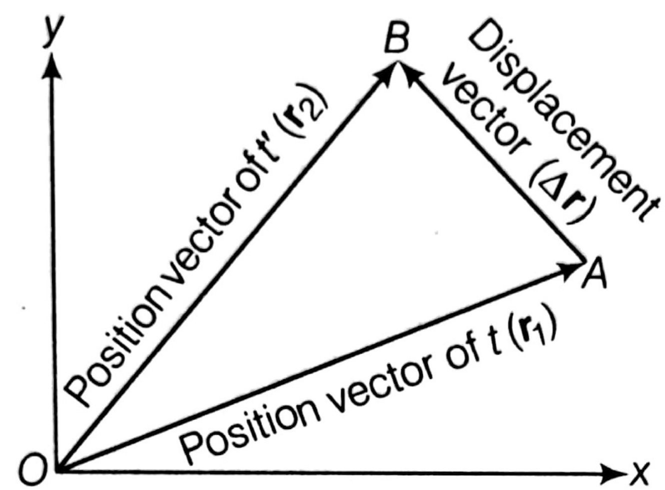


Displacement vector:

the coordinates of A be (x_1, y_1) and B be (x_2, y_2)

$$\mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$$

$$\mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$$



∴ The displacement vector for AB can be given as

$$\Delta r = \mathbf{r}_2 - \mathbf{r}_1$$

$$\boxed{\text{Displacement vector, } \Delta r = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}}}$$

Magnitude of the displacement vector is given by

$$|\Delta \mathbf{r}| = \Delta r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Delta \mathbf{r} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Multiplication of Vectors:

Case 1: A scalar multiplied with a vector

Multiplication of vector with a scalar gives again a vector

Eg:

$$\overline{\boldsymbol{P}} = \boldsymbol{m}\overline{\boldsymbol{v}} \qquad \overline{\boldsymbol{F}} = \boldsymbol{m}\overline{\boldsymbol{a}}$$

Case 2: A vector multiplied with another vector

- If resultant is a Scalar - Scalar product or Dot product

Eg: Work done $W = \mathbf{F}.\mathbf{S}$

- If resultant is a vector – Vector product or Cross product

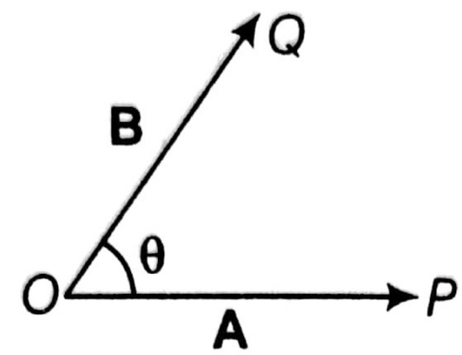
Eg: Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

SCALAR PRODUCT

Scalar product or Dot product:

It is defined as the product of the magnitudes of vectors **A** and **B** and the cosine angle between them. It is represented by

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$



When the two vectors are parallel, then $\theta = 0°$.

$$\mathbf{A} \cdot \mathbf{B} = AB\cos 0° = AB$$

When the two vectors are mutually perpendicular:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos 90^\circ = 0$$

When the two vectors are antiparallel:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos 180^\circ = -AB$$

Properties of Dot Product:

(*i*) The scalar product is commutative

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

(*ii*) The scalar product is distributive over addition

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

(*iii*) If $\vec{A}$ and $\vec{B}$ are two vectors perpendicular to each other, then their scalar product is zero.

$$\vec{A} \cdot \vec{B} = A\,B \cos 90° = 0.$$

(*iv*) If $\vec{A}$ and $\vec{B}$ are two parallel vectors having same direction, then their scalar product has the maximum positive magnitude.

$$\vec{A} \cdot \vec{B} = A\,B \cos 0° = AB$$

(*v*) The scalar product of a vector with itself is equal to the square of its magnitude.

$$\vec{A} \cdot \vec{A} = A \cdot A \cos 0° = A \cdot A = A^2 = |\vec{A}|^2$$

(*vi*) Scalar product of two similar base vectors is unity and that of two different base vectors is zero.

$$\hat{i}\,.\,\hat{i} = (1)(1)\cos 0° = 1$$

$$\hat{i}\,.\,\hat{i} = \hat{j}\,.\,\hat{j} = \hat{k}\,.\,\hat{k} = 1$$

$$\hat{i}\,.\,\hat{j} = (1)(1)\cos 90° = 0$$

$$\therefore \quad \hat{i}\,.\,\hat{j} = \hat{j}\,.\,\hat{k} = \hat{k}\,.\,\hat{i} = 0$$

Dot product between vectors **A** and **B**:

$$\vec{A}\cdot\vec{B}=\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\cdot\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)$$

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z$$

$$\hat{i}\cdot\hat{i}=1 \quad \hat{i}\cdot\hat{j}=0$$
$$\hat{j}\cdot\hat{j}=1 \quad \hat{i}\cdot\hat{k}=0$$
$$\hat{k}\cdot\hat{k}=1 \quad \hat{j}\cdot\hat{k}=0$$

Angle between the vectors:

$$\cos\theta=\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}=\frac{A_xB_x+A_yB_y+A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\sqrt{B_x^2+B_y^2+B_z^2}}$$

PROBLEM: *Find the angle between the vectors* $\vec{A} = \hat{i} + 2\hat{j} - \hat{k}$ *and* $\vec{B} = -\hat{i} + \hat{j} - 2\hat{k}$.

Solution

$$|\vec{A}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

$$|\vec{B}| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$\vec{A} \cdot \vec{B} = 1 \times (-1) + 2 \times 1 + (-1) \times (-2)$$

$$= -1 + 2 + 2 = 3$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

$$= \frac{3}{\sqrt{6} \times \sqrt{6}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

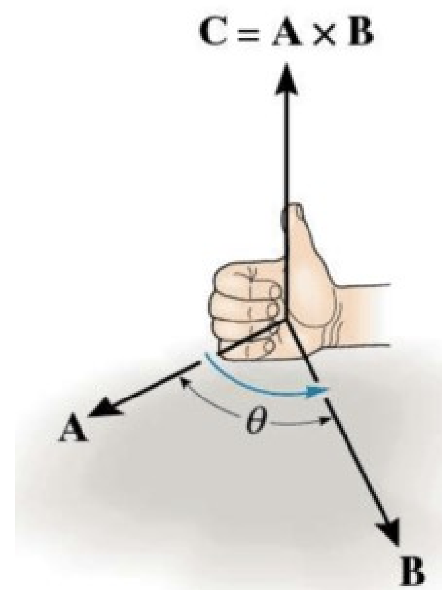
Hence $\theta = 60^\circ$

VECTOR PRODUCT

Vector product or Cross product:

It is defined as the product of the magnitude of vectors with sine of the angle between them.

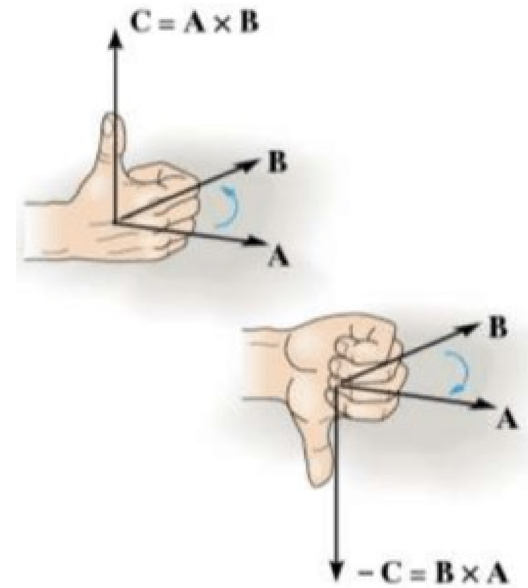
$$\bar{A} \times \bar{B} = A\,B\,\mathbf{Sin}\boldsymbol{\theta}\,\hat{n}$$



Resultant vector **C** perpendicular to the plane containing vectors **A** and **B**

Image source: https://studylib.net/

Properties of Cross Product:



(*i*) **Vector product is anti-commutative**

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(*ii*) **Vector product is distributive over addition**

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(*iii*) **Vector product of two parallel or antiparallel vectors is a null vector.**

$$\vec{A} \times \vec{B} = AB \sin (0° \text{ or } 180°)\, \hat{n} = \vec{0}$$

(*iv*) **Vector product of a vector with itself is a null vector.**

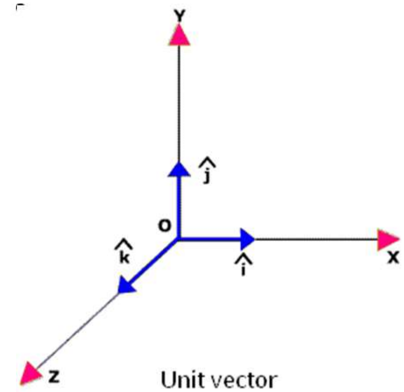
$$\vec{A} \times \vec{A} = AA \sin 0°\, \hat{n} = \vec{0}$$

(*v*) **The magnitude of the vector product of two mutually perpendicular vectors is equal to the product of their magnitudes.**

$$|\vec{A} \times \vec{B}| = AB \sin 90° = AB$$

(*vi*) **Vector product of orthogonal unit vectors.** The magnitude of each of the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ is 1 and the angle between any of two of them is 90°.

$$\therefore \quad \hat{i} \times \hat{j} = (1)(1) \sin 90° \hat{n} = \hat{n} = \hat{k}$$

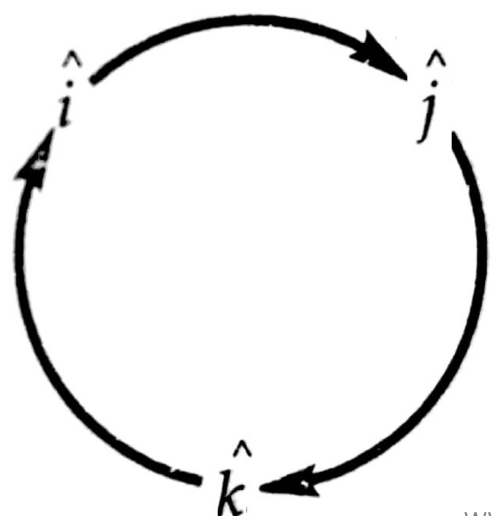


Unit vector

$$\therefore\ \hat{i}\times\hat{j}=\hat{k}$$

$$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$$

$$\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j}$$



(*vii*) **The vector product of two vectors can be expressed in terms of their rectangular components as a determinant.**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Calculate the area of the parallelogram whose two adjacent sides are formed by the vectors A = 3 i + 4 j and B = - 3 i + 7 j

Solution

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -3 & 7 & 0 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 4 & 0 \\ 7 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 4 \\ -3 & 7 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(21+12) = 33\,\hat{k}$$

Area of parallelogram

$$= |\vec{A} \times \vec{B}| = \sqrt{0^2 + 0^2 + 33^2}$$

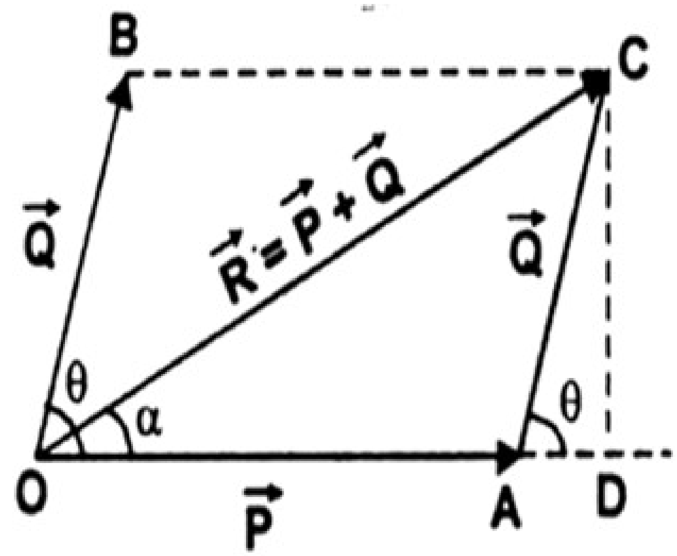
$= \mathbf{33}$ **sq. units**

PARALLELOGRAM LAW OF VECTORS

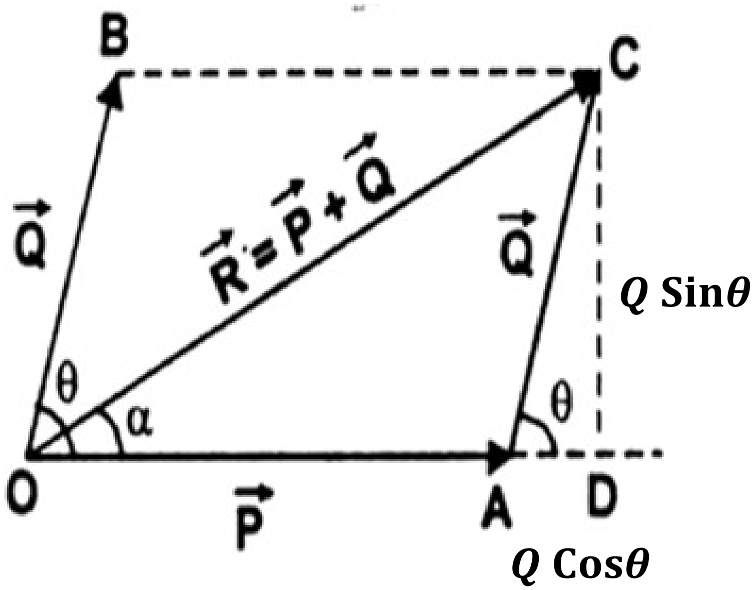
Resultant of two vectors or Parallelogram Law of Vectors:

Statement:

If two **vectors** are acting simultaneously at a point, then it can be represented both in magnitude and direction by the adjacent sides drawn from a point, according to the **parallelogram law**, the side OC of the **parallelogram** represents the resultant **vector** R.



Resultant of two vectors or Parallelogram Law of Vectors:



In right angled ΔCOD,

$$OC^2 = OD^2 + CD^2$$

$$= (OA + AD)^2 + CD^2$$

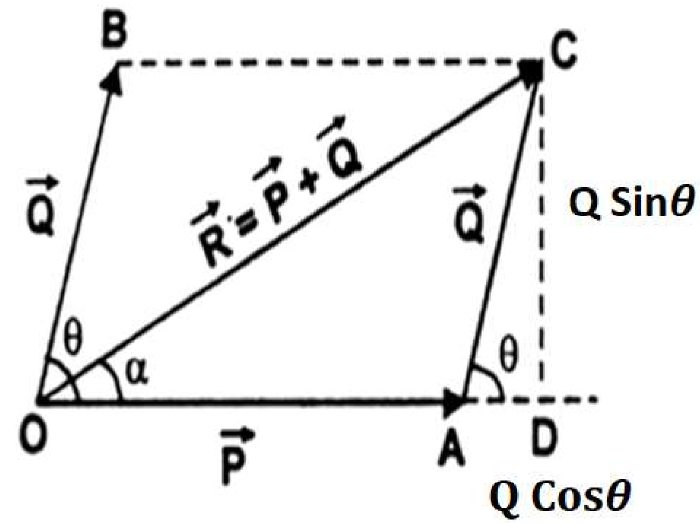
$$= (OA^2 + AD^2 + 2OA.AD) + CD^2$$

$$= P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$

$$OC^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$OC^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$\therefore \quad R = OC = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$



Let α be the angle which the resultant $\vec{R}$ makes with $\vec{P}$, then from ΔCOD,

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA + AD} = \frac{Q\sin\theta}{P + Q\cos\theta}$$

RESULTANT OF TWO VECTORS PROBLEMS

When two right angled vectors of magnitude 7 units and 24 units combine, what is the magnitude of their resultant?

Solution:

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

Given $\overline{P} = 7$ units; $\overline{Q} = 24$ units; $\theta = 90°$

$$\therefore \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = \sqrt{P^2 + Q^2}$$

$$\therefore R = \sqrt{P^2 + Q^2} = \sqrt{7^2 + 24^2} = 25 \text{ units}$$

If P = 2i + 4j + 14k and Q = 4i + 4j + 10k, find the magnitude of P + Q

$$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$$

Solution:

Given $\bar{P} = 2\bar{i} + 4\bar{j} + 14\bar{k}$ and $\bar{Q} = 4\bar{i} + 4\bar{j} + 10\bar{k}$

$$\bar{P} + \bar{Q} = 6\bar{i} + 8\bar{j} + 24\bar{k}$$

Magnitude of $|\bar{P} + \bar{Q}| = \sqrt{6^2 + 8^2 + 24^2} = \sqrt{36 + 64 + 576}$

$$= \sqrt{676} = 26$$

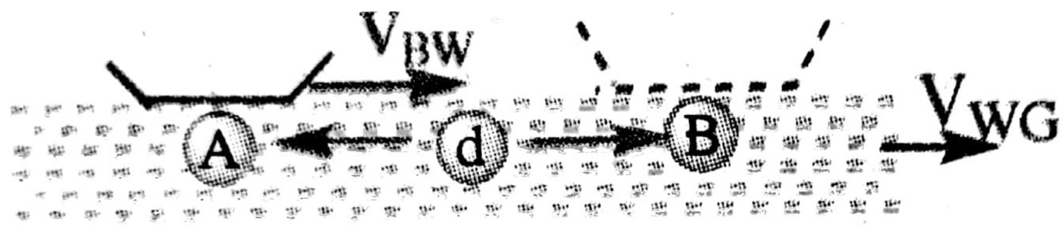
RELATIVE VELOCITY

Relative velocity:

If a boat is moving in a river flow:

Down Stream

$$V_{BG} = V_{BW} + V_{WG}$$



Time taken by the boat to travel a distance d along the river flow (from point A to B)

$$t_1 = \frac{d}{V_{BW} + V_{WG}}$$

Upstream:

$$V_{BG} = V_{BW} - V_{WG}$$

Time taken by the boat to travel a distance d opposite to the river flow (from point A to B)

$$t_2 = \frac{d}{V_{BW} - V_{WG}}$$

Ratio of time taken in Down stream to Upstream

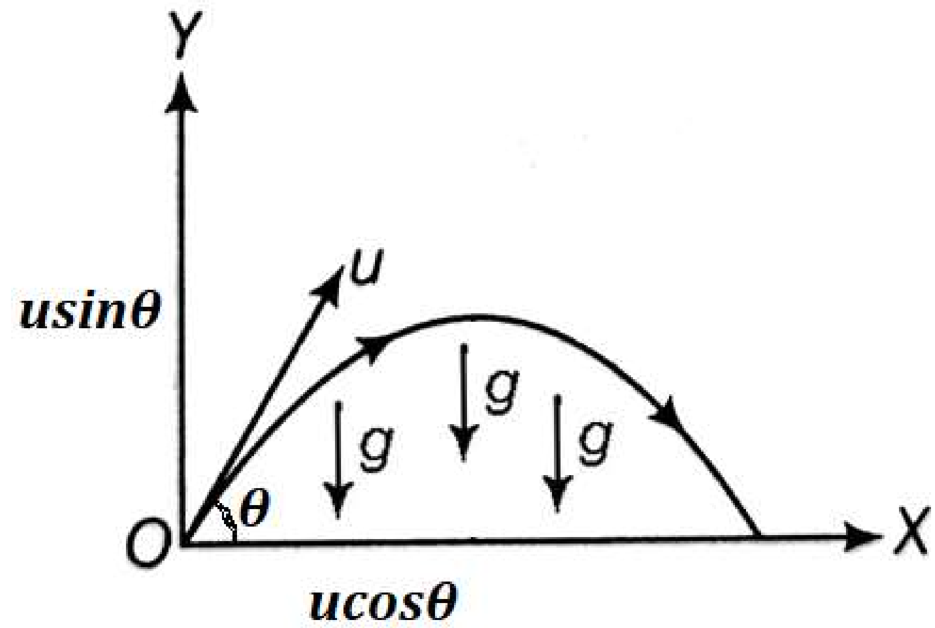
$$\frac{t_1}{t_2} = \frac{V_{BW} - V_{WG}}{V_{BW} + V_{WG}}$$

Total time of travel $T = t_d + t_u$

$$T = \frac{d}{V_{BW} + V_{WG}} + \frac{d}{V_{BW} - V_{WG}}$$

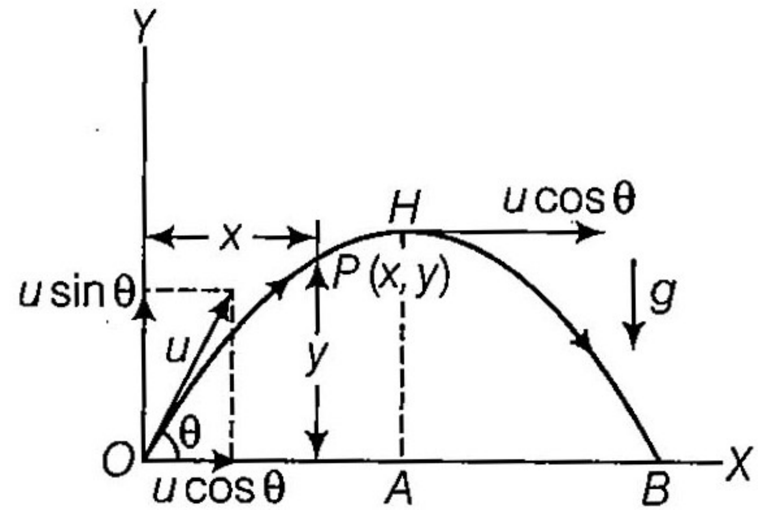
PATH OF THE PROJECTILE IS A PARABOLA

Path of the Projectile is a Parabola:



Path of the Projectile is a Parabola:

Horizontal distance travelled by the projectile in time t



$$x = u\cos\theta\, t$$

$$\therefore \quad t = \frac{x}{u\cos\theta} \quad \ldots \text{(i)}$$

Vertical distance travelled by the projectile in time t

$$y = u_y t + \frac{1}{2} a_y t^2$$

$$y = u_y t + \frac{1}{2} a_y t^2$$

But $u_y = u \sin\theta, a_y = -g$

So, $y = u \sin\theta \, t + \frac{1}{2}(-g) t^2$

$$= u \sin\theta \, t - \frac{1}{2} g t^2 \quad \dots \text{(ii)}$$

Substitute equation (i) in equation (ii), we get

$$y = u \sin\theta \left(\frac{x}{u \cos\theta} \right) - \frac{1}{2} g \left(\frac{x}{u \cos\theta} \right)^2$$

$$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{1g}{2}\left(\frac{x}{u\cos\theta}\right)^2$$

$$Let\ A = \tan\theta \qquad and \qquad B = \frac{g}{2u^2\cos\theta}$$

$$\boldsymbol{y = Ax - Bx^2}$$

This equation represents a parabola. Hence path of the projectile is a parabola.

TIME OF FLIGHT

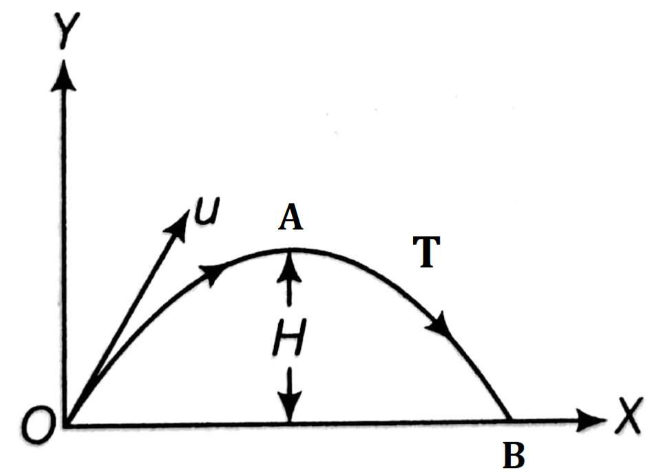
Time of flight:

The total time for which projectile is in flight i.e. time during the motion of projectile from **O** and **B**. It is denoted by **T**.

Total time of flight T

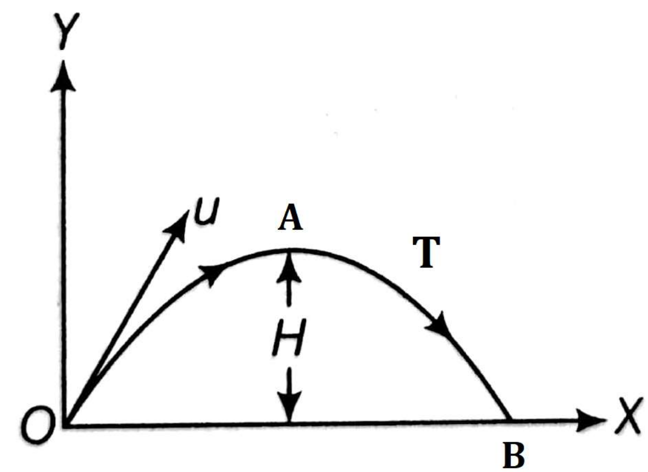
= time of ascent + time of descent

$$T = t + t = 2t \Rightarrow t = \frac{T}{2}$$



For a projectile, time of ascent equals time of descent.

$$v_y = u_y + a_y t$$



$$u_y = u\sin\theta,\ a_y = -g,\ t = \frac{T}{2} \text{ and } v_y = 0$$

$$\Rightarrow\ 0 = u\sin\theta - g\frac{T}{2}$$

$$T = \frac{2u\sin\theta}{g} \quad = \frac{2u_y}{g}$$

Time of ascent = Time of descent

$$T = \frac{2u\sin\theta}{g}$$

$$t = \frac{T}{2} = \frac{u\sin\theta}{g}$$

MAXIMUM HEIGHT AND RANGE

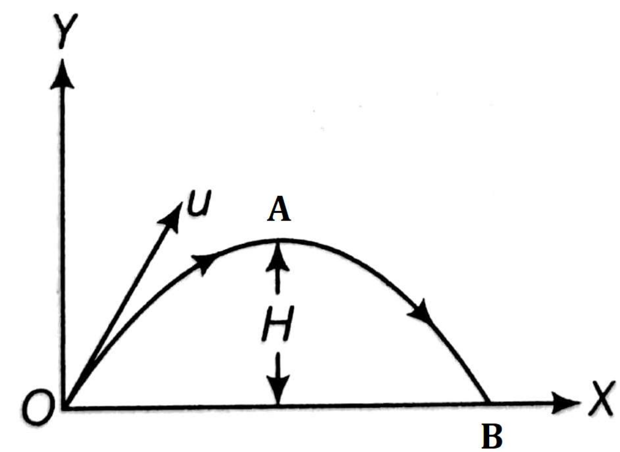
Maximum Height (H or H_{max}):

It is defined as the maximum vertical height attained by an object above the point of projection during its flight. It is denoted by H or H_{max}

$$y = u_y t + \frac{1}{2} a_y t^2$$

$$u_y = u \sin\theta,\ a_y = -g,\ y = H,\ t = \frac{T}{2} = \frac{u \sin\theta}{g}$$

$$H = u \sin\theta \frac{u \sin\theta}{g} + \frac{1}{2}(-g)\left(\frac{u \sin\theta}{g}\right)^2$$



$$H = u\sin\theta\,\frac{u\sin\theta}{g} + \frac{1}{2}(-g)\left(\frac{u\sin\theta}{g}\right)^2$$

$$= \frac{u^2}{g}\sin^2\theta - \frac{1}{2}\,\frac{u^2\sin^2\theta}{g}$$

$$H = \frac{u^2\sin^2\theta}{2g}$$

Horizontal range of a projectile:

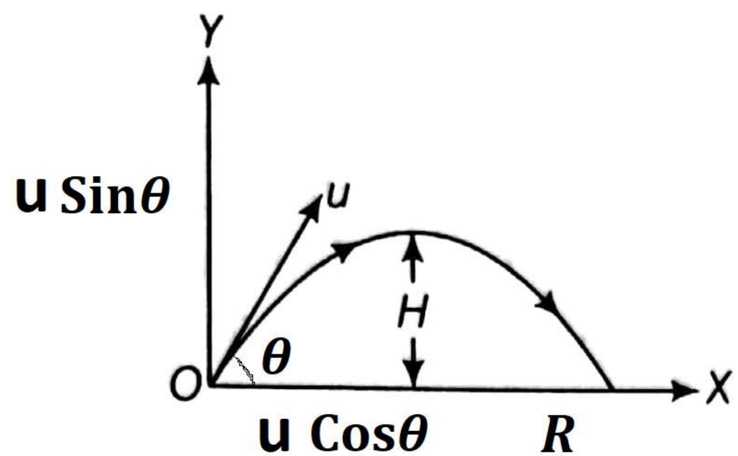
It is defined as the horizontal distance covered by the projectile during its time of flight. It is denoted by R.

Horizontal distance covered = horizontal velocity x time

$$\therefore \quad R = u\cos\theta \times T$$

$$= u\cos\theta \times 2u\frac{\sin\theta}{g}$$

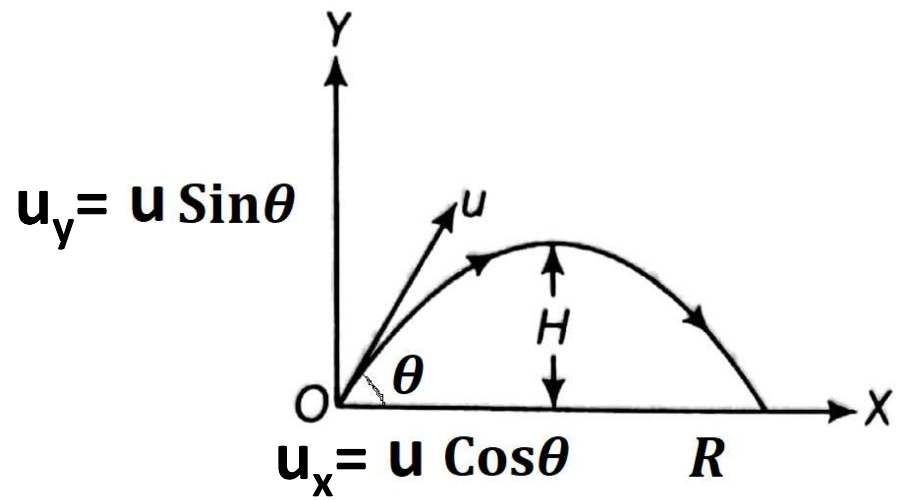
$$R = \frac{u^2 \sin 2\theta}{g}$$



$$\therefore \quad R = u\cos\theta \times T$$

$$= u\cos\theta \times 2u\frac{\sin\theta}{g}$$

$$= \frac{2u_x u_y}{g}$$



Relation between H_{max}, R and T

If θ is the angle of projection, R the range, h the maximum height, T the time of flight, then show that (a) tan θ = 4h/R and (b) $h = gT^2/8$

(a) Given angle of projection = θ

Range, $R = \dfrac{u^2 \sin 2\theta}{g}$ $h = h_{max} = \dfrac{u^2 \sin^2 \theta}{g}$ Time of flight $T = \dfrac{2u \sin\theta}{g}$

$$\frac{h}{R} = \frac{u^2 \sin^2 \theta}{2g} \times \frac{g}{u^2 \sin 2\theta}$$

$$= \frac{\sin^2 \theta}{2 \sin 2\theta} = \frac{\sin\theta \sin\theta}{2 \times 2 \sin\theta \cos\theta}$$

$$\therefore \frac{h}{R} = \frac{\tan\theta}{4}$$

$$\Rightarrow \tan\theta = \frac{4h}{R}$$

(b) $h = gT^2 / 8$

$$h = \frac{u^2 \sin^2 \theta}{2g} \qquad T = \frac{2u \sin\theta}{g} \Rightarrow \frac{u \sin\theta}{g} = T/2$$

$$\therefore h = \frac{u^2 \sin^2 \theta}{2 \cdot g^2} \cdot g = \left(\frac{T}{2}\right)^2 \times \frac{1}{2}$$

$$= \frac{gT^2}{8}$$

UNIFORM CIRCULAR MOTION

Uniform Circular motion:

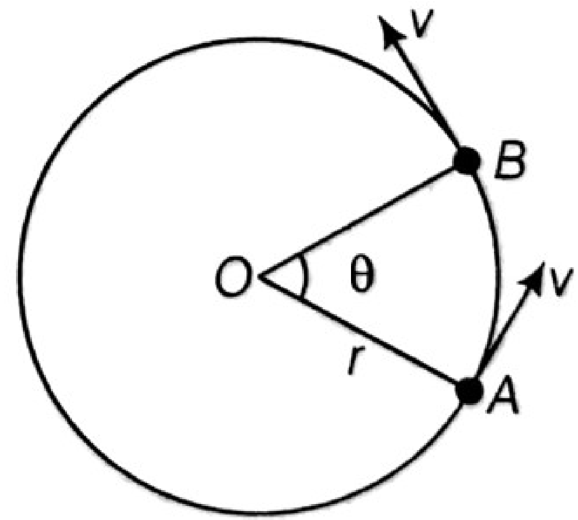
Relation between angular velocity, frequency and time period:

If time $t = T$, $\theta = 2\pi$ radian

$$\text{Angular velocity, } \omega = \frac{\theta}{t}$$

$$= \frac{2\pi}{T}$$

$$= 2\pi f$$

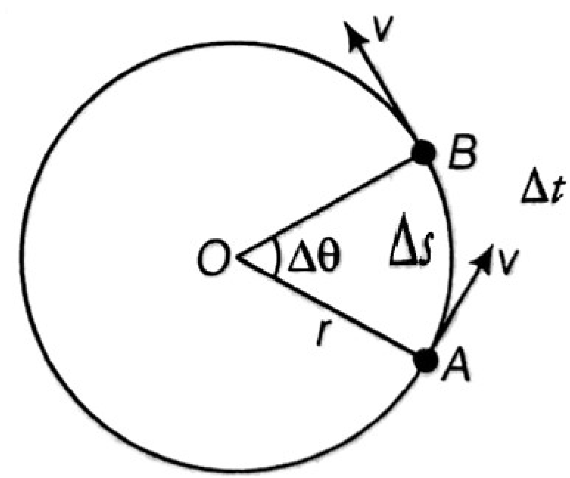


Relation between angular velocity and linear velocity:

Angular velocity $\omega = \frac{\Delta\theta}{\Delta t}$ ----- 1

Linear velocity $v = \frac{\Delta s}{\Delta t}$ ----- 2

$\Delta s = r\,\Delta\theta$ ----- 3



Substituting eqn 3 in eqn 2 $v = r\frac{\Delta\theta}{\Delta t} = r\,\omega$ From eqn 1

THANK YOU

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